STRONG SOLUTIONS TO THE COMPRESSIBLE NAVIER-STOKES-VLASOV-FOKKER-PLANCK EQUATIONS: GLOBAL EXISTENCE NEAR THE EQUILIBRIUM AND LARGE TIME BEHAVIOR

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Abstract. A kinetic-fluid model describing the evolutions of disperse two-phase flows is considered. The model consists of the Vlasov-Fokker-Planck equation for the particles (disperse phase) coupled with the compressible Navier-Stokes equations for the fluid (fluid phase) through the friction force. The friction force depends on the density, which is different from many previous studies on kinetic-fluid models and is more physical in modeling but significantly more difficult in analysis. New approach and techniques are introduced to deal with the strong coupling of the fluid and the particles. The global well-posedness of strong solution in the three-dimensional whole space is established when the initial data is a small perturbation of some given equilibrium. Moreover, the algebraic rate of convergence of solution toward the equilibrium state is obtained. For the periodic domain the same global well-posedness result still holds while the convergence rate is exponential.

1. Introduction

1.1. **The model.** Kinetic-fluid models are widely used in the description of the dynamics of disperse two-phase flows. In these two-phase flows, the disperse phase is usually considered from the statistical point of view (kinetic equation) while the dense phase is from the hydrodynamic one (fluid equations). The kinetic equation is coupled with the fluid equations through the friction force. Typical applications of two-phase flows include the dynamics of sprays [1, 2], diesel engines [32, 33, 35, 36], pollution settling processes [8], rain formation [18], wastewater treatment [5], biomedical flows [2], combustion theory [36], and so on.

In this paper we are concerned with the following system of partial differential equations (see [11]) of fluid-particle flows:

$$\partial_t F + v\cdot\nabla_x F = n\nabla_v\cdot[(v-u)F+\nabla_v F], \tag{1.1}$$

$$\partial_t n + \nabla\cdot(nu) = 0, \tag{1.2}$$

$$\partial_t(nu) + \nabla\cdot(nu\otimes u) - \Delta u + \nabla p = n\int_{\mathbb{R}^3}(v-u)F\,\mathrm{d}v, \tag{1.3}$$

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with the initial data

$$(F, n, u)|_{t=0} = (F_0(x, v), n_0(x), u_0(x)). \tag{1.4}$$

Here, the unknowns are $F = F(t, x, v) \geq 0$ for $(t, x, v) \in \mathbb{R}^+ \times \Omega \times \mathbb{R}^3$, denoting the density distribution function of particles in the phase space; and $n = n(t, x) \geq 0, u = u(t, x) \in \mathbb{R}^3$ for $(t, x) \in \mathbb{R}^+ \times \Omega$, denoting the mass density and the velocity field respectively. The pressure function p depends only on n satisfying $p'(\cdot) > 0$. In our present work, we take $p(n) = c_0 n^\gamma$ with the constants $\gamma \geq 1$ and $c_0 > 0$ for simplicity. The spatial domain is $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$ (a periodic domain in $\mathbb{R}^3$). Compared with the model introduced in [11], here we have normalized the physical constants to be one for simplicity and added the viscous term $-\Delta u$ in the momentum equation (1.3).

It is easy to check that for smooth solutions of the compressible Navier-Stokes-Vlasov-Fokker-Planck system (1.1)-(1.3), the following quantities are conserved/ dissipated:

- mass conservation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\Omega \times \mathbb{R}^3} F \,\mathrm{d}x\mathrm{d}v = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega n \,\mathrm{d}x = 0,$$

- momentum conservation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \Big\{ \int_\Omega nu \,\mathrm{d}x + \iint_{\Omega \times \mathbb{R}^3} vF \,\mathrm{d}x\mathrm{d}v \Big\} = 0,$$

- and energy/entropy dissipation:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \Big\{ n\Big(\frac{|u|^2}{2} + A\Big) + \int_{\mathbb{R}^3} \Big(F \ln F + \frac{|v|^2}{2} F\Big) \mathrm{d}v \Big\} \mathrm{d}x + \int_\Omega |\nabla u|^2 \,\mathrm{d}x \\ = - \iint_{\Omega \times \mathbb{R}^3} n \frac{|(v - u)F - \nabla_v F|^2}{F} \,\mathrm{d}x\mathrm{d}v \end{aligned} \tag{1.5}$$

with $A = \int^n \frac{p(\eta)}{\eta^2} \mathrm{d}\eta$.

Set

$$M = M(v) = \frac{1}{(2\pi)^{3/2}} \exp\Big\{ -\frac{|v|^2}{2} \Big\}.$$

From the energy/entropy dissipation (1.5), we know that $(F, n, u) \equiv (M, 1, 0)$ is an equilibrium of the system (1.1)-(1.3). Thus it is natural to introduce the transforms

$$F = M + \sqrt{M} f, \quad n = 1 + \rho \tag{1.6}$$

to rewrite the system (1.1)-(1.3) as

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f + u \cdot \nabla_v f - \frac{1}{2} u \cdot v f - u \cdot v \sqrt{M} \\ = \mathcal{L} f + \rho \Big(\mathcal{L} f - u \cdot \nabla_v f + \frac{1}{2} u \cdot v f + u \cdot v \sqrt{M} \Big), \end{aligned} \tag{1.7}$$

$$\partial_t \rho + u \cdot \nabla \rho + (1 + \rho) \mathrm{div}\, u = 0, \tag{1.8}$$

$$\partial_t u + u \cdot \nabla u + \frac{p'(1 + \rho)}{1 + \rho} \nabla \rho = \frac{1}{1 + \rho} \Delta u - u(1 + a) + b. \tag{1.9}$$

Correspondingly, the initial data (1.4) becomes

$$(f,n,u)|_{t=0}=(f_0(x,v),\rho_0(x),u_0(x))=\Big(\frac{F_0-M}{\sqrt{M}},n_0(x)-1,u_0(x)\Big). \tag{1.10}$$

In (1.7)-(1.9), $\mathcal{L}$ is the linearized Fokker-Planck operator defined by

$$\mathcal{L}f=\frac{1}{\sqrt{M}}\nabla_v\cdot\Big[M\nabla_v\big(\frac{f}{\sqrt{M}}\big)\Big],$$

and $a=a^f, b=b^f$, depending on f, are the moments of f defined by

$$a^f(t,x)=\int_{\mathbb{R}^3}\sqrt{M}f(t,x,v)\,\mathrm{d}v,\quad b^f(t,x)=\int_{\mathbb{R}^3}v\sqrt{M}f(t,x,v)\,\mathrm{d}v.$$

1.2. **Notations.** Let $\nu(v)=1+|v|^2$ and denote $|\cdot|_\nu$ by

$$|g|_\nu^2:=\int_{\mathbb{R}^3}\big\{|\nabla_v g(v)|^2+\nu(v)|g(v)|^2\big\}\,\mathrm{d}v,\quad g=g(v).$$

We use $\langle\cdot,\cdot\rangle$ to denote the inner product over the Hilbert space L^2_v, i.e.,

$$\langle g,h\rangle:=\int_{\mathbf{R}^3}g(v)h(v)\,\mathrm{d}v,\quad g,h\in L^2_v.$$

For simplicity, we shall use $\|\cdot\|$ to denote the norm of L^2_x or $L^2_{x,v}$ when there is no confusion. Define

$$\|g\|_\nu^2:=\iint_{\Omega\times\mathbb{R}^3}[|\nabla_v g(x,v)|^2+\nu(v)|g(x,v)|^2]\,\mathrm{d}x\mathrm{d}v,\quad g=g(x,v).$$

For multi-indices $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ and $\beta=(\beta_1,\beta_2,\beta_3)$, we denote by

$$\partial^\alpha_\beta\equiv\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}\partial^{\beta_1}_{v_1}\partial^{\beta_2}_{v_2}\partial^{\beta_3}_{v_3}$$

the partial derivatives with respect to $x=(x_1,x_2,x_3)$ and $v=(v_1,v_2,v_3)$. The length of α and β are defined as $|\alpha|=\alpha_1+\alpha_2+\alpha_3$ and $|\beta|=\beta_1+\beta_2+\beta_3$. We shall use the following norms:

$$\|g\|_{H^s}:=\sum_{|\alpha|\le s}\|\partial^\alpha g\|,\quad \|g\|_{H^s_{x,v}}:=\sum_{|\alpha|+|\beta|\le s}\|\partial^\alpha_\beta g\|.$$

For $g(t,x,v)$, we decompose it as the sum of the fluid part $\mathbf{P}g$ and the particle part $\{\mathbf{I}-\mathbf{P}\}g$:

$$g=\mathbf{P}g+\{\mathbf{I}-\mathbf{P}\}g. \tag{1.11}$$

Here the projection operator $\mathbf{P}$ is defined as follows:

$$\mathbf{P}:\ L^2\to\mathrm{Span}\big\{\sqrt{M},v_1\sqrt{M},v_2\sqrt{M},v_3\sqrt{M}\big\},$$

and

$$\mathbf{P}:=\mathbf{P}_0\oplus\mathbf{P}_1,\quad \mathbf{P}_0f:=a^f\sqrt{M},\quad \mathbf{P}_1f:=b^f\cdot v\sqrt{M}.$$

This macro-micro decomposition is initiated by Guo [24] for the Boltzmann equation and later introduced in [16] to study the Fokker-Planck type equations. Notice that the operator $\mathcal{L}$ satisfies

$$-\int_{\mathbb{R}^3}g\mathcal{L}g\,\mathrm{d}v\ge\lambda_0|\{\mathbf{I}-\mathbf{P}_0\}g|_\nu^2,\quad \forall\, g=g(v),$$

for some positive constant $\lambda_0 > 0$. Meanwhile, $\mathcal{L}g$ can be computed as

$$\mathcal{L}g = \mathcal{L}\{\mathbf{I}-\mathbf{P}\}g + \mathcal{L}\mathbf{P}g = \mathcal{L}\{\mathbf{I}-\mathbf{P}\}g - \mathbf{P}_1 g.$$

Therefore, we have

$$\langle -\mathcal{L}\{\mathbf{I}-\mathbf{P}\}g, g\rangle \geq \lambda_0 |\{\mathbf{I}-\mathbf{P}\}g|_\nu^2, \quad \langle -\mathcal{L}g, g\rangle \geq \lambda_0 |\{\mathbf{I}-\mathbf{P}\}g|_\nu^2 + |b^g|^2. \tag{1.12}$$

For brevity, we define the temporal energy functional and the corresponding dissipation rate for $\big(f(t,x,v), \rho(t,x), u(t,x)\big)$ as the following:

$$\begin{aligned}
\mathcal{E}_0(t) :=& \sum_{|\alpha|\leq 3} \sum_{i,j} \int_{\mathbb{R}^3} \partial_x^\alpha (\partial_{x_j} b_i + \partial_{x_i} b_j) \partial_x^\alpha \Gamma_{i,j}(\{\mathbf{I}-\mathbf{P}\}f) \,\mathrm{d}x \\
& - \sum_{|\alpha|\leq 3} \int_{\mathbb{R}^3} \partial_x^\alpha a \partial_x^\alpha \nabla_x \cdot b \,\mathrm{d}x, \qquad (1.13)\\
\mathcal{E}_1(t) :=& \|f\|^2 + \|\rho\|^2 + \|u\|^2 + \sum_{1\leq|\alpha|\leq 4} \left\{ \|\partial^\alpha f\|^2 + \left\| \frac{\sqrt{p'(1+\rho)}}{1+\rho} \partial^\alpha \rho \right\|^2 + \|\partial^\alpha u\|^2 \right\} \\
& + \tau_1 \mathcal{E}_0(t) + \tau_2 \sum_{|\alpha|\leq 3} \int_{\mathbb{R}^3} \partial^\alpha u \cdot \partial^\alpha \nabla \rho \,\mathrm{d}x, \qquad (1.14)\\
\mathcal{D}_1(t) :=& \|\nabla(a,b,\rho,u)\|_{H^3}^2 + \|b-u\|_{H^4}^2 \\
& + \sum_{|\alpha|\leq 4} \big(\|\partial^\alpha \{\mathbf{I}-\mathbf{P}\}f\|_\nu^2 + \|\partial^\alpha \nabla u\|^2 \big), \qquad (1.15)\\
\mathcal{E}_2(t) :=& \sum_{1\leq k\leq 4} C_k \sum_{\substack{|\beta|=k \\ |\alpha|+|\beta|\leq 4}} \|\partial_\beta^\alpha \{\mathbf{I}-\mathbf{P}\}f\|^2, \qquad (1.16)\\
\mathcal{D}_2(t) :=& \sum_{\substack{1\leq|\beta|\leq 4 \\ |\alpha|+|\beta|\leq 4}} \|\partial_\beta^\alpha \{\mathbf{I}-\mathbf{P}\}f\|_\nu^2, \qquad (1.17)\\
\mathcal{E}(t) :=& \mathcal{E}_1(t) + \tau_3 \mathcal{E}_2(t), \qquad (1.18)\\
\mathcal{D}(t) :=& \mathcal{D}_1(t) + \tau_3 \mathcal{D}_2(t), \qquad (1.19)
\end{aligned}$$

where $\tau_1, \tau_2, \tau_3, C_k (1\leq k\leq 4)$ are suitable constants to be chosen later. In addition, in torus we know the Poincaré inequality is true, thus, the total dissipation rate is slightly different from $\mathcal{D}(t)$. We note that

$$\begin{aligned}
\mathcal{D}_{\mathbb{T},1}(t) :=& \mathcal{D}_1(t) + \tau_4(\|a\|_{L^2}^2 + \|\rho\|_{L^2}^2) + \tau_5 \|b+u\|_{L^2}^2, \\
\mathcal{D}_{\mathbb{T}}(t) :=& \mathcal{D}_{\mathbb{T},1}(t) + \tau_3 \mathcal{D}_2(t),
\end{aligned}$$

where τ_4 and τ_5 are sufficiently small to be chosen later. Throughout this paper the letter C denotes a positive (generally large) constant and λ a positive (generally small) canstant, where both C and λ may change from line to line. The symbol $A \sim B$ means $\frac{1}{C} A \leq B \leq CA$ for some constant $C > 0$.

1.3. **Main results.** Our aim is to establish the global well-posedness and large-time behavior of strong solutions when the norm of the initial data $\|f_0\|_{H^4_{x,v}} + \|(\rho_0, u_0)\|_{H^4}$ is sufficiently small (near the equilibrium). We also obtain the different time-decay rates depending on the spatial domain $\mathbb{R}^3$ or $\mathbb{T}^3$. We now state the main results as follows.

Theorem 1.1. *Let $\Omega = \mathbb{R}^3$ and (f_0, ρ_0, u_0) be the initial data. Suppose that $F_0 = M + \sqrt{M} f_0 \geq 0$, and there exists a constant $\epsilon_0 > 0$ such that $\|f_0\|_{H^4_{x,v}} + \|(\rho_0, u_0)\|_{H^4} < \epsilon_0$. Then, the Cauchy problem* (1.7)-(1.10) *admits a unique global solution (f, ρ, u) satisfying $F = M + \sqrt{M} f \geq 0$ and*

$$f \in C([0,\infty); H^4(\mathbb{R}^3 \times \mathbb{R}^3)); \quad \rho,\, u \in C([0,\infty); H^4(\mathbb{R}^3));$$
$$\sup_{t\geq 0}(\|f(t)\|_{H^4_{x,v}} + \|(\rho, u)(t)\|_{H^4}) \leq C\big(\|f_0\|_{H^4_{x,v}} + \|(\rho_0, u_0)\|_{H^4}\big),$$

for some constant $C > 0$. Moreover, if we further assume that

$$\|f_t(0)\|^2_{H^3_{x,v}} + \|\rho_t(0)\|^2_{H^3} + \|u_t(0)\|^2_{H^3} < +\infty, \tag{1.20}$$

then

$$\sup_{x\in\mathbb{R}^3} \left\{ \sum_{|\alpha|+|\beta|\leq 1} \|\partial^\alpha_\beta f\|^2_{L^2_v} \right\} + \|\rho\|^2_{L^\infty} + \|u\|^2_{L^\infty} \leq C(1+t)^{-\frac{1}{2}} \tag{1.21}$$

for some constant $C > 0$ and all $t \geq 0$.

Remark 1.1. In the assumption (1.20), $f_t(0)$ is indeed defined through the Vlasov-Fokker-Planck equation (1.7) as follows:

$$\begin{aligned} f_t(0) :=& - v \cdot \nabla_x f_0 - u_0 \cdot \nabla_v f_0 + \frac{1}{2} u_0 \cdot v f_0 + u_0 \cdot v\sqrt{M} \\ &+ \mathcal{L} f_0 + \rho\Big(\mathcal{L} f_0 - u_0 \cdot \nabla_v f_0 + \frac{1}{2} u_0 \cdot v f_0 + u_0 \cdot v\sqrt{M}\Big); \end{aligned}$$

and $\rho_t(0)$ and $u_t(0)$ are defined similarly.

Theorem 1.2. *Let $\Omega = \mathbb{T}^3$ and (f_0, ρ_0, u_0) be the initial data. Suppose that $F_0 = M + \sqrt{M} f_0 \geq 0$, and there exists a constant $\epsilon_0 > 0$ such that $\|f_0\|_{H^4_{x,v}} + \|(\rho_0, u_0)\|_{H^4} < \epsilon_0$, and*

$$\int_{\mathbb{T}^3} a_0 \,\mathrm{d}x = 0, \quad \int_{\mathbb{T}^3} \rho_0 \,\mathrm{d}x = 0, \quad \int_{\mathbb{T}^3} (b_0 + (1+\rho_0) u_0) \,\mathrm{d}x = 0,$$

where $a_0 = \int_{\mathbb{T}^3} \sqrt{M} f_0(x, v)\,\mathrm{d}v$ and $b_0 = \int_{\mathbb{T}^3} v\sqrt{M} f_0(x, v)\,\mathrm{d}v$. Then, the Cauchy problem (1.7)-(1.10) *admits a unique global solution (f, ρ, u) satisfying $F = M + \sqrt{M} f \geq 0$ and*

$$f \in C([0,\infty); H^4(\mathbb{T}^3 \times \mathbb{R}^3)); \quad \rho,\, u \in C([0,\infty); H^4(\mathbb{T}^3));$$
$$\|f(t)\|_{H^4_{x,v}} + \|(\rho, u)(t)\|_{H^4} \leq C(\|f_0\|_{H^4_{x,v}} + \|\rho_0, u_0\|_{H^4}) e^{-\lambda t},$$

for some constant $\lambda > 0$ and any $t \geq 0$.

We remark that in the papers [12, 16, 17] for the related systems the time-decay rates are optimal in the whole space case. Here our time-decay rate $(1+t)^{-\frac{1}{2}}$ in whole space case for the solution of (1.7)-(1.9) is not optimal. The main reason is that, due to the strong coupling of the nonlinear terms in (1.7)-(1.9), the spectral analysis cannot be carried out directly. We thus present another way to obtain the decay rate. In the periodic case, by the Poincaré inequality we obtain the exponential decay which coincides with those in [12, 16, 17] in some sense.

1.4. **Some known results on kinetic-fluid models.** There exist many versions or variants of kinetic-fluid models, depending on the physical regimes under consideration, such as the compressibility of the fluid, viscosity of the fluid, species of particles, interactions between the fluid and particles, motion of the particles, and so on. Below we review some kinetic-fluid models related to our system (1.1)-(1.3). We discuss the case that both the fluid and the particle phases are isothermal. For the kinetic-fluid models with energy exchange involved, we refer the reader to [6, 22]. Generally speaking, the kinetic-fluid models can be divided into two categories: incompressible models and compressible models.

1.4.1. *Incompressible kinetic-fluid models.* If we assume that the fluid is incompressible, we obtain the incompressible kinetic-fluid models. The mathematical analysis of incompressible kinetic-fluid models has received much attention recently. In [27], Hamdache established global existence and large-time behavior of solutions for the Vlasov-Stokes system. Boudin, Desvillettes, Grandmont and Moussa [7] proved the global existence of weak solutions to the incompressible Vlasov-Navier-Stokes system on a periodic domain. Later, this result was extended to a bounded domain by Yu [37]. Goudon, He, Moussa and Zhang [19] established the global existence of classical solutions near the equilibrium for the incompressible Navier-Stokes-Vlasov-Fokker-Planck system, meanwhile Carrillo, Duan and Moussa [12] studied the corresponding inviscid case. Chae, Kang and Lee [13] obtained the global existence of weak and classical solutions for the Navier-Stokes-Vlasov-Fokker-Planck equations in a torus. Benjelloun, Desvillettes and Moussa [4] obtained the existence of global weak solutions to the incompressible Vlasov-Navier-Stokes system with a fragmentation kernel. Goudon, Jabin and Vasseur [20, 21] investigated the hydrodynamic limits to the incompressible Vlasov-Navier-Stokes system by means of some scaling and convergence methods.

Assume that the fluid is incompressible and inhomogeneous, Wang and Yu [34] obtained the global weak solution to the Navier-Stokes-Vlasov equations, while Goudon, Jin, Liu and Yan [23] presented some numerical analysis on this model.

1.4.2. *Compressible kinetic-fluid models.* Mellet and Vasseur [30, 31] studied the following compressible Navier-Stokes-Vlasov-Fokker-Planck system:

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + \operatorname{div}_v(F_d f - \nabla_v f) = 0, \\ \partial_t n + \nabla_x\cdot(nu) = 0, \\ \partial_t(nu) + \nabla_x\cdot(nu\otimes u) - \Delta u + \nabla_x p = -\displaystyle\int_{\mathbb{R}^3} F_d f\,\mathrm{d}v, \end{cases} \tag{1.22}$$

where

$$F_d = F_0(u - v), \quad F_0 > 0 \text{ a constant.} \tag{1.23}$$

In [30] they obtained the global existence of weak solutions of (1.22), and in [31] they studied the asymptotic analysis of the solutions. In [14], Chae, Kang and Lee studied the existence of the global classical solutions close to an equilibrium and obtained exponential decay of the system to the system (1.22).

When the viscous term $-\Delta u$ in the system (1.22) is dropped, Duan and Liu [17] studied the global well-posedenss of small solution in the perturbation framework. Carrillo and Goudon [10] investigated the dissipative quantities, equilibria and their stability properties. Morevoer, they also studied some asymptotic problems and the derivation of macroscopic two-phase models.

As pointed out in [31], the choice of drag force (1.23) may not be the most relevant one from a physical point of view. It could be more relevant from a physical point of view to assume that F_d depends on the density of the fluid, such as

$$F_d = n(u - v). \tag{1.24}$$

To our best knowledge, the first rigorous mathematical result concerning the case of the drag force depending on the density n was obtained by Baranger and Desvillettes [3], where the following inviscid system:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + n \nabla_v \cdot (f(u - v)) = 0, \\ \partial_t n + \nabla_x \cdot (nu) = 0, \\ \partial_t (nu) + \nabla_x \cdot (nu \otimes u) + \nabla_x p = -n \displaystyle\int_{\mathbb{R}^3} f(u - v)\, \mathrm{d}v \end{cases} \tag{1.25}$$

was considered and the local-in-time classical solutions were given. In [11], the model (1.1)-(1.3) was introduced but no mathematical result was presented. In this paper, we study the global existence and large-time behavior of solutions to the model (1.1)-(1.3). Our result is the first one on the global existence of the strong solution to the kinetic-fluid model when the drag force depends on the density.

1.5. **Strategy of the proofs of our main results.** Compared with the studies on the compressible kinetic-fluid models in literature, the analysis of the system (1.1)-(1.3) or (1.7)-(1.9) is more complicated and difficult in mathematics, as explained below.

Our approach is different from those used to obtain the global existence of classical solutions in [12, 13, 19] on the incompressible kinetic-fluid models and [14, 17] on the compressible kinetic-fluid models where the drag force is independent of the density. Due to the nonlinear terms caused by the strong coupling of f and ρ in the system (1.7)-(1.9), we cannot use the existing results on the Vlasov-Fokker-Planck system to obtain the regularity of f. Thus, we have to deal with the derivative of the particle velocity v in our arguments.

Our main difficulties in obtaining the large time behavior of solutions come again from the strong coupling of f and ρ in the system (1.7)-(1.9). It prevents us from taking the advantage of the linearized spectral analysis to gain the rate

of convergence of solutions as in [12–14, 17, 19]. To overcome these difficulties, we shall construct some novel functionals and adopt with modification some techniques in [15] to deal with the compressible Navier-Stokes equations and in [28] for the Landau equation. Through the detailed analysis on the strong coupling terms of f and ρ we obtain the desired estimates. We believe that the methods developed in this paper can be applied to study the more complicated models in [6, 22], which is our forthcoming research project.

The rest of paper is organized as follows. In Section 2, we shall establish the global existence of classical solutions to the problem (1.7)-(1.10) in the spatial domain $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$. By the fine energy estimates we mainly use the local existence of strong solutions and continuum argument, motivated by [25, 26] for the Boltzmann and Landau equations. In Section 3, by means of the energy estimates for the temporal derivative of the system (1.7)-(1.9) and the Gronwall-type inequality, we eventually obtain the large time behavior. In Section 4, we shall establish the uniform a priori estimates with the aid of some energy functionals and corresponding dissipation rate. Although the uniform a priori estimates obtained in Section 4 are needed in Sections 2 and 3, we shall present this lengthy part in the last section of the paper for the convenience of readers.

In the rest of this paper, we shall omit the integral domain $\Omega \times \mathbb{R}^3$ or $\mathbb{R}^3$ in the integrals for simplicity.

2. Global existence of the classical solutions

In this section, we shall establish the global existence of classical solutions to the problem (1.7)-(1.10) in the spatial domain $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$. It is well known that by the uniform a priori estimates we shall obtain the global existence of solutions with the help of the local existence as well as the continuum argument, under the smallness and regularity conditions on the initial data. Here we first construct the iteration process to obtain the unique local solution, then the global existence of solutions follows from the continuum argument and the uniform a priori estimates obtained in Section 4.

Now we define iteratively the sequence $(F^n, \rho^n, u^n)_{n=0}^{\infty}$ as the solutions to the system:

$$
\begin{cases}
\partial_t F^{n+1} + v \cdot \nabla_x F^{n+1} - (1+\rho^n)\nabla_v \big(vF^{n+1} + \nabla_v F^{n+1}\big) \\
\qquad\qquad\qquad = -(1+\rho^n)u^n \cdot \nabla_v F^{n+1}, \\
\partial_t \rho^{n+1} + u^n \cdot \nabla \rho^{n+1} + (1+\rho^n)\operatorname{div} u^n = 0, \\
\partial_t u^{n+1} - \dfrac{1}{1+\rho^n}\Delta u^n = -u^n \cdot \nabla u^n + \gamma(1+\rho^n)^{\gamma-2}\nabla\rho^n + u^n(1+a^n) + b^n.
\end{cases}
$$

Setting $F^n = M + \sqrt{M} f^n$, we can rewrite the above system as

$$
\begin{aligned}
&\partial_t f^{n+1} + v \cdot \nabla_x f^{n+1} - (1+\rho^n)\mathcal{L} f^{n+1} \\
&\qquad\qquad = -(1+\rho^n)u^n \cdot \Big(\nabla_v f^{n+1} - \frac{v}{2} f^{n+1} - v\sqrt{M}\Big),
\end{aligned}
\tag{2.1}
$$

$$\partial_t \rho^{n+1} + u^n \cdot \nabla \rho^{n+1} + (1+\rho^n)\operatorname{div} u^{n+1} = 0, \tag{2.2}$$

$$\partial_t u^{n+1} - \frac{1}{1+\rho^n}\Delta u^{n+1} = -u^n \cdot \nabla u^n + \gamma(1+\rho^n)^{\gamma-2}\nabla\rho^n + u^n(1+a^n) + b^n, \tag{2.3}$$

where $n = 0, 1, 2, \ldots$, and $(f^0, \rho^0, u^0) = (f_0, \rho_0, u_0)$ is the starting value of iteration.

We define the solution space $X(0,T;A)$ by

$$X(0,T;A) := \left\{ \begin{array}{c} f \in C([0,T], H^4(\Omega\times\mathbb{R}^3)),\ M+\sqrt{M}f \geq 0; \\ \rho \in C([0,T], H^4(\Omega)) \bigcap C^1([0,T], H^3(\Omega)); \\ u \in C([0,T], H^4(\Omega)) \bigcap C^1([0,T], H^2(\Omega)); \\ \sup_{0\leq t\leq T}\{\|f(t)\|_{H^4_{x,v}} + \|(\rho,u)\|_{H^4}\} \leq A; \\ \rho_1 = \frac{1}{2}(-1+\inf\rho(0,x)) > -1; \\ \inf_{0\leq t\leq T, x\in\Omega}\rho(t,x) \geq \rho_1. \end{array} \right\} \tag{2.4}$$

The main result of this section reads as follows.

Theorem 2.1. *There exist $A_0 > 0$ and $T^* > 0$, such that if $f_0 \in H^4(\Omega\times\mathbb{R}^3), \rho_0 \in H^4(\Omega), u_0 \in H^4(\Omega)$ with $F_0 = M + \sqrt{M}f_0 \geq 0$ and $\mathcal{E}(0) \leq \frac{A_0}{2}$, with $\mathcal{E}(0) \sim \|f_0\|^2_{H^4_{x,v}} + \|(\rho_0,u_0)\|^2_{H^4}$ (see the specific definition of $\mathcal{E}(0)$ in Section 4), then for each $n \geq 1$, (f^n, ρ^n, u^n) is well-defined with*

$$(f^n, \rho^n, u^n) \in X(0, T^*; A_0). \tag{2.5}$$

Moveover, the following statements hold:

(1) *$(f^n, \rho^n, u^n)_{n\geq 0}$ is a Cauchy sequence in the Banach space $C([0,T^*]; H^3(\Omega\times\mathbb{R}^3)) \times C\big([0,T^*], H^3(\Omega)\big) \times C\big([0,T^*], H^3(\Omega)\big)$,*
(2) *the corresponding limit function denoted by (f,ρ,u) belong to $X(0,T^*;A_0)$,*
(3) *(f,ρ,u) are solutions to the Cauchy problem* (1.7)-(1.10)*,*
(4) *(f,ρ,u) is unique in $X(0,T^*;A_0)$ for the problem* (1.7)-(1.10).

Proof. Let $T^* > 0$ be a constant which will be fixed later. For brevity, we can assume that (f^n, ρ^n, u^n) are smooth enough in order to take the forthcoming calculations, otherwise, we can consider the following regularized iteration system:

$$\begin{aligned}
&\partial_t f^{n+1,\epsilon} + v\cdot\nabla_x f^{n+1,\epsilon} - (1+\rho^{n,\epsilon})\mathcal{L}f^{n+1,\epsilon} \\
&\qquad\qquad = -(1+\rho^{n,\epsilon})u^{n,\epsilon}\cdot\Big(\nabla_v f^{n+1,\epsilon} - \frac{v}{2}f^{n+1,\epsilon} - v\sqrt{M}\Big), \\
&\partial_t\rho^{n+1,\epsilon} + u^{n,\epsilon}\cdot\nabla\rho^{n+1,\epsilon} + (1+\rho^{n,\epsilon})\operatorname{div} u^{n+1,\epsilon} = 0, \\
&\partial_t u^{n+1,\epsilon} - \frac{1}{1+\rho^{n,\epsilon}}\Delta u^{n+1,\epsilon} = -u^{n,\epsilon}\cdot\nabla u^{n,\epsilon} \\
&\qquad\qquad + \gamma(1+\rho^{n,\epsilon})^{\gamma-2}\nabla\rho^{n,\epsilon} + u^{n,\epsilon}(1+a^{n,\epsilon}) + b^{n,\epsilon}, \\
&(f^{n+1,\epsilon}, \rho^{n+1,\epsilon}, u^{n+1,\epsilon})(0) = (f_0^\epsilon, \rho_0^\epsilon, u_0^\epsilon)
\end{aligned}$$

for any $\epsilon > 0$ with $(f_0^\epsilon, \rho_0^\epsilon, u_0^\epsilon)$ a smooth approximation of (f_0, ρ_0, u_0) and pass to the limit by letting $\epsilon \to 0$.

Applying ∂_x^α with $|\alpha|\leq 4$ to the equation (2.1), multiplying the result by $\partial_x^\alpha f^{n+1}$ and then taking integration over Ω, one has

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^\alpha f^{n+1}\|^2+\int(1+\rho^n)\langle-\mathcal{L}\partial^\alpha f^{n+1},\partial^\alpha f^{n+1}\rangle\,\mathrm{d}x\\
&=\sum_{0\langle\beta\leq\alpha}C_{\alpha,\beta}\int\partial^\beta\rho^n\langle\mathcal{L}\partial^{\alpha-\beta}f^{n+1},\partial^\alpha f^{n+1}\rangle\,\mathrm{d}x\\
&\quad-\iint\partial^\alpha\Big\{(1+\rho^n)u^nM^{-\frac12}\nabla_v(M+\sqrt{M}f^{n+1})\Big\}\partial^\alpha f^{n+1}\,\mathrm{d}x\mathrm{d}v\\
&\leq C(1+\|\rho^n\|_{H^4})\|u^n\|_{H^4}\|f^{n+1}\|_{L^2_v(H^4)}\\
&\quad+C(1+\|\rho^n\|_{H^4})\|u^n\|_{H^4}\|f^{n+1}\|_{L^2_v(H^4)}\|\partial^\alpha f^{n+1}\|_\nu\\
&\quad+C\|\rho^n\|_{H^4}\Big(\sum_{|\alpha'|\leq3}\|\partial^{\alpha'f^{n+1}}\|_\nu\Big)\|\partial^\alpha f^{n+1}\|_\nu.
\end{aligned}\tag{2.6}$$

Notice that

$$\int(1+\rho^n)\langle-\mathcal{L}\partial^\alpha f^{n+1},\partial^\alpha f^{n+1}\rangle\,\mathrm{d}x\geq\lambda\|\{\mathbf{I}-\mathbf{P}_0\}\partial^\alpha f^{n+1}\|_\nu^2.$$

By adding $\|\mathbf{P}_0\partial^\alpha f^{n+1}\|_\nu^2$ to both sides on the inequality (2.6), then, taking summation over $|\alpha|\leq 4$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\alpha|\leq4}\|\partial^\alpha f^{n+1}\|^2+\lambda\sum_{|\alpha|\leq4}\|\partial^\alpha f^{n+1}\|_\nu^2\\
&\leq C(1+\|\rho^n\|_{H^4})\|u^n\|_{H^4}\|f^{n+1}\|_{L^2_v(H^4)}\\
&\quad+C(1+\|\rho^n\|_{H^4}^2)\|u^n\|_{H^4}^2\|f^{n+1}\|_{L^2_v(H^4)}^2\\
&\quad+C\|\rho^n\|_{H^4}\sum_{|\alpha|\leq4}\|\partial^\alpha f^{n+1}\|_\nu^2+C\|f^{n+1}\|_{L^2_v(H^4)}^2.
\end{aligned}$$

Similarly, for any $0\leq t\leq T\leq T^*$, we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f^{n+1}\|_{H^4_{x,v}}^2+\lambda\sum_{|\alpha|+|\beta|\leq4}\|\partial_\beta^\alpha f^{n+1}\|_\nu^2\\
&\quad\leq C(1+\|\rho^n\|_{H^4}^2)\|f^{n+1}\|_{H^4_{x,v}}^2+C\|\rho^n\|_{H^4}\sum_{|\alpha|+|\beta|\leq4}\|\partial_\beta^\alpha f^{n+1}\|_\nu^2\\
&\qquad+C(1+\|\rho^n\|_{H^4}^2)\|u^n\|_{H^4}^2(1+\|f^{n+1}\|_{H^4_{x,v}}^2).
\end{aligned}\tag{2.7}$$

Next, according to [29], for the system (2.2) and (2.3), there exists a unique solution (ρ^{n+1},u^{n+1}) satisfying $\rho^{n+1}\geq\rho_1$, and

$$\begin{aligned}
&\rho^{n+1}\in C([0,T],H^4(\Omega))\cap C^1([0,T],H^3(\Omega)),\\
&u^{n+1}\in C([0,T],H^4(\Omega))\cap C^1([0,T],H^2(\Omega)).
\end{aligned}$$

Now we estimate $\frac{\mathrm{d}}{\mathrm{d}t}\|(\rho^{n+1},u^{n+1})\|_{H^4}^2$. Applying ∂^α ($|\alpha|\leq4$) to the system (2.2) and (2.3), multiplying the results by $\partial^\alpha\rho^{n+1}$ and $\partial^\alpha u^{n+1}$ respectively, and then taking integration and summation, one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\rho^{n+1}\|_{H^4}^2+\|u^{n+1}\|_{H^4}^2\Big)+\lambda\sum_{|\alpha|\leq4}\int|\nabla\partial^\alpha u^{n+1}|^2\mathrm{d}x$$

$$
\begin{aligned}
&\le C(1+\|\rho^n\|_{H^4}^2+\|u^n\|_{H^4}^2)\|\rho^{n+1}\|_{H^4}^2+C(1+\|\rho^n\|_{H^4}^2)\|u^{n+1}\|_{H^4}^2\\
&\quad +C(\|u^n\|_{H^4}^2+\|f^n\|_{H^4}^2)(1+\|u^n\|_{H^4}^2)+C\|\rho^n\|_{H^4}^2(1+\|\rho^n\|_{H^4}^6). \qquad (2.8)
\end{aligned}
$$

Adding up (2.7) and (2.8) gives

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|f^{n+1}\|_{H^4_{x,v}}^2+\|\rho^{n+1}\|_{H^4}^2+\|u^{n+1}\|_{H^4}^2\Big)\\
&\quad +\lambda\sum_{|\alpha|+|\beta|\le 4}\|\partial^\alpha_\beta f^{n+1}\|_\nu^2+\lambda\sum_{|\alpha|\le 4}\|\nabla\partial^\alpha u^{n+1}\|^2\\
&\le C(1+\|\rho^n\|_{H^4}^2)\|f^{n+1}\|_{H^4_{x,v}}^2+C(1+\|\rho^n\|_{H^4}^2)\|u^n\|_{H^4}^2(1+\|f^{n+1}\|_{H^4_{x,v}}^2)\\
&\quad +C(1+\|\rho^n\|_{H^4}^2+\|u^n\|_{H^4}^2)\|\rho^{n+1}\|_{H^4}^2+C(1+\|\rho^n\|_{H^4}^2)\|u^{n+1}\|_{H^4}^2\\
&\quad +C(1+\|u^n\|_{H^4}^2)(\|f^n\|_{H^4_{x,v}}^2+\|u^n\|_{H^4}^2)+C\|\rho^n\|_{H^4}^2(1+\|\rho^n\|_{H^4}^6)\\
&\quad +C\|\rho^n\|_{H^4}\sum_{|\alpha|+|\beta|\le 4}\|\partial^\alpha_\beta f^{n+1}\|_\nu^2. \qquad (2.9)
\end{aligned}
$$

Using induction, we may assume $A_n(T)\le A_0$ and $A_n(0)\le\frac{A_0}{2}$ for some $A_0>0$ with

$$
A_n(T):=\sup_{0\le t\le T}\big\{\|\rho^n(t)\|_{H^4}^2+\|u^n(t)\|_{H^4}^2+\|f^n(t)\|_{H^4_{x,v}}^2\big\}.
$$

Integrating (2.9) over $[0,T]$ gives

$$
\begin{aligned}
&A_{n+1}(T)+\lambda\int_0^T\Big\{\sum_{|\alpha|\le 4}\|\nabla\partial^\alpha u^{n+1}\|^2+\sum_{|\alpha|+|\beta|\le 4}\|\partial^\alpha_\beta f^{n+1}\|_\nu^2\Big\}\mathrm{d}t\\
&\le A_{n+1}(0)+C(1+A_n^{\frac12}(T)+A_n^2(T))A_{n+1}(T)T+C(A_n(T)+A_n^4(T))T\\
&\quad +CA_n^{\frac12}(T)\sum_{|\alpha|+|\beta|\le 4}\|\partial^\alpha_\beta f^{n+1}\|_\nu^2\\
&\le \frac{A_0}{2}+C(1+A_0^2)TA_{n+1}(T)+C(A_0+A_0^4)T\\
&\quad +CA_n^{\frac12}(T)\sum_{|\alpha|+|\beta|\le 4}\|\partial^\alpha_\beta f^{n+1}\|_\nu^2. \qquad (2.10)
\end{aligned}
$$

It follows that, for $T\le T^*$,

$$
\big(1-C(1+A_0^2)T\big)A_{n+1}(T)\le\frac{A_0}{2}+C(A_0+A_0^4)T.
$$

Choosing T^* satisfying $T^*\le\frac{A_0}{2}$, and A_0 sufficiently small, we conclude that

$$
A_{n+1}\le A_0.
$$

For the equation of F^{n+1}, with the help of the maximum principle, we have

$$
F^{n+1}=M+\sqrt{M}f^{n+1}\ge 0.
$$

Now we explain that $\|f^{n+1}\|_{H^4_{x,v}}^2$ is continuous over $0\le t\le T^*$. In fact, it follows from the inequality:

$$
\Big|\|f^{n+1}(t)\|_{H^4_{x,v}}^2-\|f^{n+1}(s)\|_{H^4_{x,v}}^2\Big|
$$

$$
\begin{aligned}
&= \Big| \int_s^t \frac{\mathrm{d}}{\mathrm{d}\eta} \|f^{n+1}(\eta)\|^2_{H^4_{x,v}} \mathrm{d}\eta \Big| \\
&\leq C A_0^{\frac{1}{2}} \sum_{|\alpha|+|\beta|\leq 4} \int_s^t \|\partial^\alpha_\beta f^{n+1}\|^2_\nu \mathrm{d}\eta + C(A_0 + A_0^3)|t-s|,
\end{aligned} \tag{2.11}
$$

which can be proved by the same process as the proof of (2.7). Meanwhile, $\|\partial^\alpha_\beta f^{n+1}\|^2_\nu$ is integrable over $[0, T^*]$. Hence, (2.5) holds true for $n+1$ and so it does for any $n \geq 0$.

Next, we study the following system:

$$
\begin{aligned}
&\partial_t(f^{n+1} - f^n) + v \cdot \nabla_x(f^{n+1} - f^n) - L(f^{n+1} - f^n) \\
&\quad = \rho^n \mathcal{L} f^{n+1} + (1+\rho^n) u^n (v\sqrt{M} + \frac{v}{2} f^{n+1} - \nabla_v f^{n+1}) \\
&\qquad - (1+\rho^{n-1}) u^{n-1} (v\sqrt{M} + \frac{v}{2} f^n - \nabla_v f^n) - \rho^{n-1} \mathcal{L} f^n, \\
&\partial_t(u^{n+1} - u^n) - \frac{1}{1+\rho^n} \Delta(u^{n+1} - u^n) \\
&\quad = \Big(\frac{1}{1+\rho^n} - \frac{1}{1+\rho^{n-1}}\Big) \Delta u^n \\
&\qquad - \big((u^n - u^{n-1}) \cdot \nabla u^n + u^{n-1} \cdot \nabla(u^n - u^{n-1})\big) \\
&\qquad + \gamma\big((1+\rho^n)^{\gamma-2} \nabla\rho^n - (1+\rho^{n-1})^{\gamma-2} \nabla\rho^{n-1}\big) \\
&\qquad + b^n - b^{n-1} + (a^n - a^{n-1}) u^{n-1} + (u^n - u^{n-1})(1 + a^n), \\
&\partial_t(\rho^{n+1} - \rho^n) + u^n \cdot \nabla(\rho^{n+1} - \rho^n) \\
&\quad = -(u^n - u^{n-1}) \cdot \nabla\rho^n - (1+\rho^n)\, \mathrm{d}v(u^{n+1} - u^n) - (\rho^n - \rho^{n-1}) \operatorname{div} u^n.
\end{aligned}
$$

Similarly to (2.9), we obtain that

$$
\begin{aligned}
&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \big\{ \|f^{n+1} - f^n\|^2_{H^3_{x,v}} + \|\rho^{n+1} - \rho^n\|^2_{H^3} + \|u^{n+1} - u^n\|^2_{H^3} \big\} \\
&\quad + \lambda \sum_{|\alpha|+|\beta|\leq 3} \|\partial^\alpha_\beta (f^{n+1} - f^n)\|^2_\nu + \lambda \|\nabla(u^{n+1} - u^n)\|^2_{H^3} \\
&\leq C\big(1 + \|\rho^n\|^4_{H^4} + \|u^n\|^4_{H^4}\big) \\
&\quad \times \big(\|f^{n+1} - f^n\|^2_{H^3_{x,v}} + \|\rho^{n+1} - \rho^n\|^2_{H^3} + \|u^{n+1} - u^n\|^2_{H^3}\big) \\
&\quad + C\|\rho^n\|_{H^3} \sum_{|\alpha|+|\beta|\leq 3} \|\partial^\alpha_\beta (f^{n+1} - f^n)\|^2_\nu + C(1 + \|u^{n-1}\|^2_{H^3}) \|f^n - f^{n-1}\|^2_{H^3} \\
&\quad + C\big\{(1 + \|\rho^n\|^2_{H^4})(1 + \|f^n\|^2_{H^3}) + \|(u^{n-1}, f^n, u^n)\|^2_{H^4}\big\} \|u^n - u^{n-1}\|^2_{H^3} \\
&\quad + C\bigg\{ 1 + \|(u^n, u^{n-1})\|^2_{H^4} + \|(\rho^n, u^n)\|^2_{H^4} \|\rho^n\|^6_{H^4} + (1 + \|u^{n-1}\|^2_{H^3}) \|f^n\|^2_{H^3} \\
&\quad + \sum_{|\alpha|+|\beta|\leq 3} \|\partial^\alpha_\beta f^n\|^2_\nu \bigg\} \|\rho^n - \rho^{n-1}\|^2_{H^3}.
\end{aligned}
$$

Here we have used the Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$. Since $\mathcal{E}(0)$, T^*, and A_0 are sufficiently small, by using (2.10), we know that

$$\sup_n \int_0^{T^*} \sum_{|\alpha|+|\beta|\le 4} \|\partial^\alpha_\beta f^n\|_\nu^2 \,\mathrm{d}s$$

is also sufficiently small. Hence, there exists a constant $\kappa < 1$, such that

$$\begin{aligned}\sup_{0<t\le T^*} &\big\{\|f^{n+1}-f^n\|_{H^4} + \|\rho^{n+1}-\rho^n\|_{H^4} + \|u^{n+1}-u^n\|_{H^4}\big\} \\ &\le \kappa \sup_{0<t\le T^*} \big\{\|f^n-f^{n-1}\|_{H^4} + \|\rho^n-\rho^{n-1}\|_{H^4} + \|u^n-u^{n-1}\|_{H^4}\big\}. \qquad (2.12)\end{aligned}$$

According to (2.12), we conclude that $(f^n,\rho^n,u^n)_{n\ge0}$ is a Cauchy sequence in the Banach space $C\big([0,T^*],H^3(\Omega\times\mathbb{R}^3)\big)\times C\big([0,T^*],H^3(\Omega)\big)\times C\big([0,T^*],H^3(\Omega)\big)$. Hence, in this Banach space, there exists a limit function (f,ρ,u) such that (f,ρ,u) is a solution to the Cauchy problem (1.7)-(1.10) by letting $n\to\infty$. From the fact that $F^n(t,x,v)\ge 0$ and the Sobolev embedding theorem, we deduce that

$$F(t,x,v)\ge 0, \qquad \sup_{0\le t\le T^*}\|f(t)\|_{H^4_{x,v}} \le A_0.$$

Similarly to the proof of (2.11), we see that $f\in C\big([0,T^*],H^3(\Omega\times\mathbb{R}^3)\big)$. Thus, we can conclude that $(f,\rho,u)\in X(0,T^*;A_0)$. Finally, let $(\bar f,\bar\rho,\bar u)\in X(0,T^*,A_0)$ be another solution to the Cauchy problem (1.7)-(1.10). By taking the similar process to that of (2.12), we have

$$\begin{aligned}\sup_{0<t\le T^*} &\big\{\|f-\bar f\|_{H^4} + \|\rho-\bar\rho\|_{H^4} + \|u-\bar u\|_{H^4}\big\} \\ &\le \kappa \sup_{0<t\le T^*} \big\{\|f-\bar f\|_{H^4} + \|\rho-\bar\rho\|_{H^4} + \|u-\bar u\|_{H^4}\big\}\end{aligned}$$

for $\kappa<1$. Hence we deduce that $f\equiv\bar f, \rho\equiv\bar\rho, u\equiv\bar u$ and uniqueness follows. $\square$

Since $\mathcal{E}(0)\sim\|f_0\|^2_{H^4_{x,v}}+\|(\rho_0,u_0)\|^2_{H^4}$, there exists $\epsilon_0>0$, such that if $\|f_0\|_{H^4_{x,v}}+\|(\rho_0,u_0)\|_{H^4}<\epsilon_0$, we have $\mathcal{E}(0)\le\frac{A_0}{2}$. Next, with the aid of Theorem 2.1, we obtain the local solution on $[0,T^*]$ that satisfies the uniform a priori estimate (4.28) in Section 4. Finally, by taking the standard bootstrap arguments similar to those in [16, 24, 29], we obtain the global existence and uniqueness of classical solutions in both Theorem 1.1 and Theorem 1.2.

3. Large time behavior of the classical solutions

In this section, we investigate the time-decay rates of global solutions to the problem (1.7)-(1.10). We obtain the algebraic rate of convergence of solution toward the equilibrium state in the whole space case, while for the periodic domain case, the convergence rate is exponential.

3.1. **The case of the whole space.** In this subsection we consider the large time behavior of classical solutions in the whole space $\Omega = \mathbb{R}^3$. In order to obtain the desired decay rate (1.21) in Theorem 1.1, we first introduce some new functionals which are similar to those [15, 28] in spirit. Then we perform the energy analysis to the temporal derivative of the system (1.7)-(1.9), instead of the original equations, to gain the one-order derivative. Finally, we combine the energy estimates together with the uniform a priori estimates obtained in Section 4 and the following Gronwall-type inequality to obtain the time decay of the solutions. For simplicity, we shall denote $\big(f_t, \rho_t, u_t\big) = \big(\partial_t f, \partial_t \rho, \partial_t u\big)$.

Lemma 3.1 ([15])**.** *Let $y(t) \in C^1([t_0, \infty))$ satisfy $y(t) \geq 0, A = \int_{t_0}^{\infty} y(s)\mathrm{d}s < +\infty$ and $y'(t) \leq a(t)y(t)$ for all $t \geq t_0$. If $a(t) \geq 0$ and $B = \int_{t_0}^{\infty} a(s)\mathrm{d}s < +\infty$, then*

$$y(t) \leq \frac{(t_0 y(t_0) + 1)\exp(A + B) - 1}{t}, \quad \forall\, t \geq t_0.$$

First of all, we consider the system for (f_t, ρ_t, u_t):

$$\begin{aligned} &\partial_t f_t + v \cdot \nabla_x f_t - (1+\rho)\mathcal{L} f_t = \rho_t \mathcal{L} f - (1+\rho) u \cdot \Big(\nabla_v f_t - \frac{v}{2} f_t\Big) \\ &\qquad - \big(\rho_t u + (1+\rho)u_t\big) \cdot \Big(\nabla_v f - \frac{v}{2} f - v\sqrt{M}\Big), \qquad (3.1) \\ &\partial_t \rho_t + u \cdot \nabla \rho_t + \rho_t \operatorname{div} u = -u_t \nabla \rho - (1+\rho) \operatorname{div} u_t, \qquad (3.2) \\ &\partial_t u_t + u_t \cdot \nabla u + u \cdot \nabla u_t - \frac{1}{1+\rho}\Delta u_t + \frac{p'(1+\rho)}{1+\rho}\nabla \rho_t + \frac{1}{(1+\rho)^2}\rho_t \Delta u \\ &\qquad = -\gamma(\gamma-2)(1+\rho)^{\gamma-3}\rho_t \nabla \rho - u_t(1+a) - u a_t + b_t. \qquad (3.3) \end{aligned}$$

Notice that there are some similar structures between the system (3.1)-(3.3) and the system (1.7)-(1.9), and we can benefit from the proofs of Lemmas 4.1-4.5. Thus, we will omit some details of the proof of the following lemma.

Proposition 3.1. *Assume that (f, ρ, u) is the solution obtained in Theorem 1.1. Then, we have*

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|f_t\|^2 + \|\rho_t\|^2 + \|u_t\|^2\big) + \lambda \|\{\mathbf{I}-\mathbf{P}\} f_t\|_\nu^2 + \lambda \|\nabla u_t\|^2 + \lambda \|u_t - b_t\|^2 \\ &\quad \leq \epsilon \|\nabla(\rho_t, b_t)\|^2 + C_\epsilon\big(\|\{\mathbf{I}-\mathbf{P}\} f\|_\nu^2 + \|u - b\|^2 + \|\nabla u\|_{H^1}^2\big)\|\rho_t\|^2 \\ &\qquad + C(1 + \|\rho\|_{H^2})\|u\|_{H^2}\|\{\mathbf{I}-\mathbf{P}\} f_t\|_\nu^2 + C(1 + \|\rho\|_{H^2}^2 + \|u\|_{H^2}^2) \\ &\qquad \times \Big\{ \sum_{|\alpha| \leq 2} \|\partial_x^\alpha \{\mathbf{I}-\mathbf{P}\} f\|_\nu^2 + \|\nabla_x(a,b)\|_{H^1}^2 + \|\nabla_x(\rho, u)\|_{H^2}^2 \Big\} \\ &\qquad \times \big(\|f_t\|^2 + \|(\rho_t, u_t)\|_{H^2}^2\big) \qquad (3.4) \end{aligned}$$

for $\epsilon > 0$ sufficiently small.

Proof. First, multiplying (3.1) by f_t, (3.2) by ρ_t, and (3.3) by u_t, respectively, and then integrating over $\mathbb{R}^3$ and taking summation, we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|f_t\|^2 + \|\rho_t\|^2 + \|u_t\|^2\big) + \lambda \|\{\mathbf{I}-\mathbf{P}\} f_t\|_\nu^2 + \lambda \|\nabla u_t\|^2 + \|u_t - b_t\|^2$$

$$
\begin{aligned}
&\le \iint \rho_t \mathcal{L} f f_t \,\mathrm{d}x\mathrm{d}v + \int \rho_t u \cdot b_t \,\mathrm{d}x - \iint \big(\rho_t u + (1+\rho)u_t\big)\cdot\Big(\nabla_v - \frac{v}{2}\Big) f f_t \,\mathrm{d}x\mathrm{d}v \\
&\quad - \int a u_t^2 \,\mathrm{d}x + \frac{1}{2}\iint (1+\rho)u\cdot v f_t^2 \,\mathrm{d}x\mathrm{d}v - \int a_t u \cdot u_t \,\mathrm{d}x + \int \rho(u_t - b_t)\cdot b_t \,\mathrm{d}x \\
&\quad - \int u_t \cdot \nabla\rho\rho_t \,\mathrm{d}x + \gamma(\gamma-2)\int (1+\rho)^{\gamma-3}\rho_t u_t \cdot \nabla\rho \,\mathrm{d}x \\
&\quad - \int (1+\rho) \operatorname{div} u_t \rho_t \,\mathrm{d}x - \int \frac{p'(1+\rho)}{1+\rho} u_t \cdot \nabla\rho_t \,\mathrm{d}x \\
&\quad - \int u\cdot\nabla\rho_t\rho_t \,\mathrm{d}x - \int \rho_t^2 \operatorname{div} u \,\mathrm{d}x - \int u_t \cdot \nabla u \cdot u_t \,\mathrm{d}x - \int u\cdot\nabla u_t \cdot u_t \,\mathrm{d}x \\
&\quad + \int \frac{1}{(1+\rho)^2}\nabla\rho\cdot\nabla u_t \cdot u_t \,\mathrm{d}x - \int \frac{1}{(1+\rho)^2}\rho_t \Delta u \cdot u_t \,\mathrm{d}x. \qquad (3.5)
\end{aligned}
$$

Next, we estimate each term on the right hand side of (3.5). For the first two terms, we rewrite them as follows:

$$
\begin{aligned}
&\iint \rho_t \mathcal{L} f f_t \,\mathrm{d}x\mathrm{d}v + \int \rho_t u \cdot b_t \,\mathrm{d}x \\
&\qquad = - \iint \rho_t \Big(\nabla_v + \frac{v}{2}\Big)\{\mathbf{I}-\mathbf{P}\} f \Big(\nabla_v + \frac{v}{2}\Big)\{\mathbf{I}-\mathbf{P}\} f_t \,\mathrm{d}x\mathrm{d}v \\
&\qquad\quad - \iint \rho_t \Big(\nabla_v + \frac{v}{2}\Big)\{\mathbf{I}-\mathbf{P}\} f \Big(\nabla_v + \frac{v}{2}\Big)\mathbf{P} f_t \,\mathrm{d}x\mathrm{d}v \\
&\qquad\quad - \iint \rho_t \Big(\nabla_v + \frac{v}{2}\Big)\mathbf{P} f \Big(\nabla_v + \frac{v}{2}\Big)\{\mathbf{I}-\mathbf{P}\} f_t \,\mathrm{d}x\mathrm{d}v \\
&\qquad\quad - \iint \rho_t \Big(\nabla_v + \frac{v}{2}\Big)\mathbf{P} f \Big(\nabla_v + \frac{v}{2}\Big)\mathbf{P} f_t \,\mathrm{d}x\mathrm{d}v + \int \rho_t u \cdot b_t \,\mathrm{d}x \\
&\qquad := \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4. \qquad (3.6)
\end{aligned}
$$

For the term Π_1, one has

$$
\begin{aligned}
\Pi_1 &\le C\|\rho_t\| \int \sum_{|\alpha|\le 2} \Big\|\Big(\nabla_v + \frac{v}{2}\Big)\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\} f\Big\|_{L_x^2} \Big\|\Big(\nabla_v + \frac{v}{2}\Big)\{\mathbf{I}-\mathbf{P}\} f_t\Big\|_{L_x^2} \mathrm{d}v \\
&\le C\|\rho_t\| \sum_{|\alpha|\le 2} \|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\} f\|_\nu \|\{\mathbf{I}-\mathbf{P}\} f_t\|_\nu \\
&\le \epsilon\|\{\mathbf{I}-\mathbf{P}\} f_t\|_\nu^2 + C_\epsilon\|\rho_t\|^2 \sum_{|\alpha|\le 2} \|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\} f\|_\nu^2.
\end{aligned}
$$

By the definition of $\mathbf{P}$ and noticing the equality $\big(\nabla_v + \frac{v}{2}\big)g = b^g\sqrt{M}$ for any g, one has

$$
\begin{aligned}
\Pi_2 &= \iint \rho_t \Big(\nabla_v + \frac{v}{2}\Big)\{\mathbf{I}-\mathbf{P}\} f \cdot b_t\sqrt{M} \,\mathrm{d}x\mathrm{d}v \\
&\le C\|\nabla b_t\|_{H^1}\|\rho_t\|\|\{\mathbf{I}-\mathbf{P}\} f\|_\nu \\
&\le \epsilon\|\nabla b_t\|_{H^1}^2 + C_\epsilon\|\{\mathbf{I}-\mathbf{P}\} f\|_\nu^2\|\rho_t\|^2
\end{aligned}
$$

for any $\epsilon > 0$. Similarly, for the terms Π_3 and Π_4, we have

$$
\Pi_3 = \iint \rho_t \Big(\nabla_v + \frac{v}{2}\Big)\{\mathbf{I}-\mathbf{P}\} f \cdot b_t\sqrt{M} \,\mathrm{d}x\mathrm{d}v
$$

$$
\begin{aligned}
&\le C\|\nabla b_t\|_{H^1}\|\rho_t\|\|\{\mathbf{I}-\mathbf{P}\}f\|_\nu\\
&\le \epsilon\|\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu^2 + C_\epsilon\|\nabla b\|_{H^1}^2\|\rho_t\|^2,\\
\Pi_4 &= \int \rho_t(u-b)\cdot b_t\,\mathrm{d}x\\
&\le C\|\nabla b_t\|_{H^1}\|u-b\|\|\rho_t\|\\
&\le \epsilon\|\nabla b_t\|_{H^1}^2 + C_\epsilon\|u-b\|^2\|\rho_t\|^2
\end{aligned}
$$

for any $\epsilon > 0$. Plugging the above estimates into (3.6), we can rewrite it as

$$
\begin{aligned}
\iint \rho_t\mathcal{L}ff_t\,\mathrm{d}x\mathrm{d}v + \int \rho_t u\cdot b_t\,\mathrm{d}x &\le \epsilon\big(\|\nabla b_t\|_{H^1}^2 + \|\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu^2\big)\\
&+ C_\epsilon\big(\sum_{|\alpha|\le 2}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2 + \|u-b\|^2 + \|\nabla b\|_{H^1}^2\big)\|\rho_t\|^2
\end{aligned}
\tag{3.7}
$$

for any $\epsilon > 0$.

For the third and fourth terms on the right hand side of (3.5), we have

$$
\begin{aligned}
&-\iint \big(\rho_t u + (1+\rho)u_t\big)\cdot\Big(\nabla_v - \frac{v}{2}\Big)ff_t\,\mathrm{d}x\mathrm{d}v - \int au_t^2\,\mathrm{d}x\\
&\quad= \int \big(\rho_t u + (1+\rho)u_t\big)f\Big(\nabla_v + \frac{v}{2}\Big)\{\mathbf{I}-\mathbf{P}\}f_t\,\mathrm{d}x\mathrm{d}v\\
&\qquad+ \int \big(\rho_t u + (1+\rho)u_t\big)f\Big(\nabla_v + \frac{v}{2}\Big)\mathbf{P}f_t\,\mathrm{d}x\mathrm{d}v - \int au_t^2\,\mathrm{d}x.
\end{aligned}
\tag{3.8}
$$

We have the following estimates for the right hand side terms in (3.8):

$$
\begin{aligned}
&\iint \big(\rho_t u + (1+\rho)u_t\big)f\Big(\nabla_v + \frac{v}{2}\Big)\{\mathbf{I}-\mathbf{P}\}f_t\,\mathrm{d}x\mathrm{d}v\\
&\le C\int \big(\|uf\|_{L_x^\infty}\|\rho_t\| + \|f\|_{L_x^\infty}(1+\|\rho\|_{L^\infty})\|u_t\|\big)\|\Big(\nabla_v + \frac{v}{2}\Big)\{\mathbf{I}-\mathbf{P}\}f_t\|\mathrm{d}v\\
&\le C(1+\|\rho\|_{H^2} + \|u\|_{H^2})\|(\rho_t,u_t)\|\|\nabla_x f\|_{L_v^2(H_x^1)}\|\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu\\
&\le \epsilon\|\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu^2 + C_\epsilon(1+\|(\rho,u)\|_{H^2}^2)\\
&\quad\times\big(\sum_{|\alpha|\le 2}\|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2 + \sum_{|\alpha|\le 1}\|\nabla_x(a,b)\|^2\big)\|(\rho_t,u_t)\|^2,\\
&\iint \big(\rho_t u + (1+\rho)u_t\big)f\Big(\nabla_v + \frac{v}{2}\Big)\mathbf{P}f_t\,\mathrm{d}x\mathrm{d}v - \int au_t^2\,\mathrm{d}x\\
&= \int \big(\rho_t u + \rho u_t\big)ab_t\,\mathrm{d}x + \int au_t\cdot(b_t-u_t)\,\mathrm{d}x\\
&\le \epsilon\|u_t-b_t\|^2 + C_\epsilon\|\nabla a\|_{H^1}^2(\|f_t\|^2 + \|u_t\|^2) + C_\epsilon\|(\nabla\rho,\nabla u)\|_{H^1}^2(\|\rho_t\|^2 + \|u_t\|^2)
\end{aligned}
$$

for any $\epsilon > 0$. Plugging the above estimates into (3.8) gives

$$
\begin{aligned}
&-\iint \big(\rho_t u + (1+\rho)u_t\big)\Big(\nabla_v - \frac{v}{2}\Big)f\cdot f_t\,\mathrm{d}x\mathrm{d}v - \int au_t^2\,\mathrm{d}x\\
&\quad\le \epsilon\big(\|u_t-b_t\|^2 + \|\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu^2\big) + C_\epsilon(1+\|\rho\|_{H^2}^2 + \|u\|_{H^2}^2)\\
&\qquad\times\Big\{\sum_{|\alpha|\le 2}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2 + \|\nabla(a,b,\rho,u)\|_{H^1}^2\Big\}\big(\|f_t\|^2 + \|(\rho_t,u_t)\|^2\big).
\end{aligned}
\tag{3.9}
$$

The remaining terms on right hand side of (3.5) can be treated easily. Hence we only list the bounds below and omit the details for brevity. We have

$$
\begin{aligned}
&\iint \frac{1}{2}(1+\rho)u\cdot v f_t^2\,\mathrm{d}x\mathrm{d}v - \int a_t u\cdot u_t\,\mathrm{d}x \\
&\qquad \le \epsilon\big(\|u_t-b_t\|^2+\|\{\mathbf{I}-\mathbf{P}\}f_t\|^2\big)+C_\epsilon\big\{(1+\|\rho\|_{H^2}^2)\|\nabla u\|_{H^1}^2 \\
&\qquad\quad +\|\nabla\rho\|_{H^1}\|\nabla u\|_{H^1}\big\}\|f_t\|^2+C(1+\|\rho\|_{H^2})\|u\|_{H^2}\|\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu^2, \quad (3.10)
\end{aligned}
$$

$$
\int \rho(u_t-b_t)\cdot b_t\,\mathrm{d}x \le \epsilon\|u_t-b_t\|^2+C_\epsilon\|\nabla\rho\|_{H^1}^2\|f_t\|^2, \tag{3.11}
$$

$$
\begin{aligned}
&\int (1+\rho)\,\mathrm{div}\,u_t\rho_t\,\mathrm{d}x+\int \frac{p'(1+\rho)}{1+\rho}u_t\cdot\nabla\rho_t\,\mathrm{d}x \\
&\qquad \le \epsilon\|\nabla u_t\|^2+C_\epsilon\|\nabla\rho\|_{H^1}^2\|\rho_t\|^2, \qquad (3.12)
\end{aligned}
$$

$$
\int u\cdot\nabla\rho_t\rho_t\,\mathrm{d}x+\int \rho_t^2\,\mathrm{div}\,u\,\mathrm{d}x \le \epsilon\|\nabla\rho_t\|^2+C_\epsilon\|\nabla u\|_{H^1}^2\|\rho_t\|^2, \tag{3.13}
$$

$$
\begin{aligned}
&\int \rho_t u_t\cdot\nabla\rho\,\mathrm{d}x-\gamma(\gamma-2)\int (1+\rho)^{\gamma-3}\rho_t u_t\cdot\nabla\rho\,\mathrm{d}x \\
&\qquad \le \epsilon\|\nabla u_t\|^2+C_\epsilon\|\nabla\rho\|_{H^1}^2\|\rho_t\|^2, \qquad (3.14)
\end{aligned}
$$

$$
\int u_t\cdot\nabla u\cdot u_t\,\mathrm{d}x+\int u\cdot\nabla u_t\cdot u_t\,\mathrm{d}x \le \epsilon\|\nabla u_t\|^2+C_\epsilon\|\nabla u\|_{H^1}^2\|u_t\|^2, \tag{3.15}
$$

$$
\begin{aligned}
&\int \frac{1}{(1+\rho)^2}\nabla\rho\cdot\nabla u_t\cdot u_t\,\mathrm{d}x-\int \frac{1}{(1+\rho)^2}\rho_t\Delta u_t\cdot u_t\,\mathrm{d}x \\
&\qquad \le \epsilon\|\nabla u_t\|^2+C_\epsilon\|(\nabla\rho,\nabla u)\|_{H^2}^2\|(\rho_t,u_t)\|^2. \qquad (3.16)
\end{aligned}
$$

Plugging all the above estimates (3.7), (3.9), and (3.10)-(3.16) into (3.5) and then choosing ϵ sufficiently small yield (3.4). □

Proposition 3.2. *Assume that (f,ρ,u) is the solution obtained in Theorem 1.1. Then, we have*

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{1\le|\alpha|\le 3}\left\{\|\partial^\alpha f_t\|^2+\left\|\frac{\sqrt{p'(1+\rho)}}{1+\rho}\partial^\alpha\rho_t\right\|^2+\|\partial^\alpha u_t\|^2\right\} \\
&\quad +\lambda\sum_{1\le|\alpha|\le 3}\big\{\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu^2+\|\partial^\alpha(u_t-b_t)\|^2+\|\nabla\partial^\alpha u_t\|^2\big\} \\
&\le \epsilon\big(\|\nabla a_t\|_{H^1}^2+\|\nabla b_t\|_{H^2}^2\big)+C_\epsilon\big(1+\|\rho\|_{H^3}^2\big)\|\nabla u\|_{H^2}^2\|f_t\|_{L_v^2(H_x^3)}^2 \\
&\quad +C_\epsilon\left\{\sum_{|\alpha|\le 3}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\|\nabla_x(a,b,\rho,u)\|_{H^2}^2\right\}\|(\rho_t,u_t)\|_{H^3}^2 \\
&\quad +C\|\rho\|_{H^3}\|\nabla(a_t,b_t)\|_{H^2}^2+C\big(1+\|\rho\|_{H^3}\big)\|u\|_{H^4}\|\nabla\rho_t\|_{H^2}^2 \\
&\quad +C\big\{(1+\|\rho\|_{H^2})\|u\|_{H^2}+\|\rho\|_{H^3}\big\}\sum_{1\le|\alpha|\le 3}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu^2 \\
&\quad +C\left\{\sum_{|\alpha|\le 3}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\|\nabla(a,b,\rho)\|_{H^2}^2+(1+\|\rho\|_{H^3}^2)\|\nabla u\|_{H^2}^2\right\}\|f_t\|_{L_v^2(H_x^3)}^2
\end{aligned}
$$

$$
\begin{aligned}
&+C\Big\{\sum_{|\alpha|\le 3}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\|\nabla(a,b,\rho,u)\|_{H^3}^2+\|\nabla u\|_{H^4}^2\\
&+\|\nabla\rho\|_{H^3}^3+\|\rho\|_{H^3}^4\|\nabla\rho\|_{H^2}^2\Big\}\|(\rho_t,u_t)\|_{H^3}^2 \qquad (3.17)
\end{aligned}
$$

for $\epsilon>0$ sufficiently small.

Proof. Applying $\partial^\alpha(1\le|\alpha|\le 3)$ to (3.1)-(3.3), multiplying the results by $\partial^\alpha f_t$, $\frac{p'(1+\rho)}{(1+\rho)^2}\partial^\alpha\rho_t$, and $\partial^\alpha u_t$ respectively, and then adding them together, we obtain

$$
\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\partial^\alpha f_t\|^2+\|\frac{\sqrt{p'(1+\rho)}}{1+\rho}\partial^\alpha\rho_t\|^2+\|\partial^\alpha u_t\|^2\Big)\\
&+\int\langle-\mathcal{L}\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f_t,\partial^\alpha f_t\rangle\,\mathrm{d}x+\|\partial^\alpha(u_t-b_t)\|^2+\int\frac{1}{1+\rho}|\nabla\partial^\alpha u_t|^2\,\mathrm{d}x\\
=&\iint\partial^\alpha\Big((\rho_t u+(1+\rho)u_t)\frac{v}{2}f\Big)\partial^\alpha f_t\,\mathrm{d}x\mathrm{d}v-\iint\partial^\alpha\big((\rho_t u+(1+\rho)u_t)\cdot\nabla_v f\big)\partial^\alpha f_t\,\mathrm{d}x\mathrm{d}v\\
&+\iint\partial^\alpha\Big((1+\rho)u\cdot\frac{v}{2}f_t\Big)\partial^\alpha f_t\,\mathrm{d}x\mathrm{d}v-\int\partial^\alpha(ua_t)\cdot\partial^\alpha u_t\,\mathrm{d}x\\
&-\int[-\partial^\alpha,(1+\rho)u\cdot\nabla_v]f_t\partial^\alpha f_t\,\mathrm{d}x+\int\partial^\alpha(\rho_t\mathcal{L}f)\partial^\alpha f_t\,\mathrm{d}x\\
&-\int\partial^\alpha(\rho\mathcal{L}f_t)\partial^\alpha f_t\,\mathrm{d}x-\int\frac{p'(1+\rho)}{(1+\rho)^2}u\cdot\nabla\partial^\alpha\rho_t\partial^\alpha\rho_t\,\mathrm{d}x\\
&+\int\partial^\alpha\big(\rho_t u+\rho u_t\big)\cdot\partial^\alpha b_t\,\mathrm{d}x+\int\partial^\alpha u_t\cdot\nabla\frac{p'(1+\rho)}{1+\rho}\partial^\alpha\rho_t\,\mathrm{d}x\\
&+\int[-\partial^\alpha,\rho\nabla_x\cdot]u_t\frac{p'(1+\rho)}{(1+\rho)^2}\partial^\alpha\rho_t\,\mathrm{d}x+\int[-\partial^\alpha,u\cdot\nabla_x]\rho_t\frac{p'(1+\rho)}{(1+\rho)^2}\partial^\alpha\rho_t\,\mathrm{d}x\\
&+\frac12\int\partial_t\big(\frac{p'(1+\rho)}{(1+\rho)^2}\big)|\partial^\alpha\rho_t|^2\,\mathrm{d}x-\int\partial^\alpha\big(\rho_t\,\operatorname{div}u+\nabla\rho\cdot u_t\big)\frac{p'(1+\rho)}{(1+\rho)^2}\partial^\alpha\rho_t\,\mathrm{d}x\\
&-\int u\cdot\nabla\partial^\alpha u_t\cdot\partial^\alpha u_t\,\mathrm{d}x-\int\nabla\frac{1}{1+\rho}\nabla\partial^\alpha u_t\cdot\partial^\alpha u_t\,\mathrm{d}x-\int\partial^\alpha(u_t\cdot\nabla u)\cdot\partial^\alpha u_t\,\mathrm{d}x\\
&-\sum_{1\le\gamma\le\alpha}C_{\alpha,\gamma}\int\partial^\gamma\big(\frac{1}{1+\rho}\big)\partial^{\alpha-\gamma}\Delta u_t\cdot\partial^\alpha u_t\,\mathrm{d}x-\int\partial^\alpha\big(\frac{1}{(1+\rho)^2}\rho_t\Delta u\big)\cdot\partial^\alpha u_t\,\mathrm{d}x\\
&+\int[-\partial^\alpha,u\cdot\nabla]u_t\cdot\partial^\alpha u_t\,\mathrm{d}x+\int\Big[-\partial^\alpha,\frac{p'(1+\rho)}{1+\rho}\nabla\Big]\rho_t\cdot\partial^\alpha u_t\,\mathrm{d}x\\
&-\gamma(\gamma-2)\int\partial^\alpha\big((1+\rho)^{\gamma-3}\rho_t\nabla\rho\big)\cdot\partial^\alpha u_t\,\mathrm{d}x-\int\partial^\alpha(au_t)\cdot\partial^\alpha u_t\,\mathrm{d}x. \qquad (3.18)
\end{aligned}
$$

Now we estiamte the terms on the right hand side of (3.18). First, we have

$$
\begin{aligned}
&\iint\partial^\alpha\big((\rho_t u+\rho u_t)\cdot\nabla_v f\big)\partial^\alpha f_t\,\mathrm{d}x\mathrm{d}v+\int\partial^\alpha\Big((\rho_t u+\rho u_t)\cdot\frac{v}{2}f\Big)\partial^\alpha f_t\,\mathrm{d}x\mathrm{d}v\\
\le&\int\big(\|\nabla u\|_{H^2}\|\rho_t\|_{H^3}+\|\nabla\rho\|_{H^2}\|u_t\|_{H^3}\big)\Big\|\Big(\nabla_v+\frac{v}{2}\Big)\nabla_x f\Big\|_{H_x^2}\|\partial^\alpha f_t\|_{L_x^2}\mathrm{d}v\\
\le&C\big(\|\nabla u\|_{H^2}\|\rho_t\|_{H^3}+\|\nabla\rho\|_{H^2}\|u_t\|_{H^3}\big)\Big\|\Big(\nabla_v+\frac{v}{2}\Big)\nabla_x f\Big\|_{L^2(H_x^2)}\|\partial^\alpha f_t\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\Big\{\sum_{|\alpha|\leq 3}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\|\nabla(a,b)\|_{H^2}^2\Big\}\|\partial^\alpha f_t\|^2+C\|(\nabla\rho,\nabla u)\|_{H^2}^2\|(\rho_t,u_t)\|_{H^3}^2,\\
&\iint \partial^\alpha\Big(u_t\Big(\nabla_v+\frac{v}{2}\Big)f\Big)\partial^\alpha f_t\,\mathrm{d}x\mathrm{d}v\\
&\leq C\|\nabla u_t\|_{H^2}\Big\|\Big(\nabla_v+\frac{v}{2}\Big)\nabla_x f\Big\|_{L^2(H_x^2)}\|\partial^\alpha f_t\|\\
&\leq \eta\|\nabla u_t\|_{H^2}^2+C_\eta\Big\{\sum_{|\alpha|\leq 3}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\|\nabla(a,b)\|_{H^2}^2\Big\}\|\partial^\alpha f_t\|^2
\end{aligned}
$$

with η small enough. Thus the first two terms on the right hand side can be bounded by

$$
\begin{aligned}
&\Big\{\sum_{|\alpha|\leq 3}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\|\nabla(a,b)\|_{H^2}^2\Big\}\|\partial^\alpha f_t\|^2+C\|(\nabla\rho,\nabla u)\|_{H^2}^2\|(\rho_t,u_t)\|_{H^3}^2\\
&\qquad+\eta\|\nabla u_t\|_{H^2}^2+C_\eta\Big\{\sum_{|\alpha|\leq 3}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\|\nabla(a,b)\|_{H^2}^2\Big\}\|\partial^\alpha f_t\|^2. \qquad (3.19)
\end{aligned}
$$

For the third and forth terms on the right hand side of (3.18), we can rewrite them as

$$
\begin{aligned}
&\iint \partial^\alpha\Big((1+\rho)u\cdot\frac{v}{2}f_t\Big)\partial^\alpha f_t\,\mathrm{d}x\mathrm{d}v-\int\partial^\alpha(ua_t)\cdot\partial^\alpha u_t\,\mathrm{d}x\\
=&\iint(1+\rho)u\cdot\frac{v}{2}\partial^\alpha f_t\partial^\alpha f_t\,\mathrm{d}x\mathrm{d}v-\int\partial^\alpha a_t u\cdot\partial^\alpha u_t\,\mathrm{d}x\\
&+\sum_{1\leq\gamma\leq\alpha}C_{\alpha,\gamma}\Big(\iint\partial^\gamma\big((1+\rho)u\big)\cdot\frac{v}{2}\partial^{\alpha-\gamma}f_t\partial^\alpha f_t\,\mathrm{d}x\mathrm{d}v-\int\partial^{\alpha-\gamma}a_t\partial^\gamma u\cdot\partial^\alpha u_t\,\mathrm{d}x\Big)\\
=&\frac{1}{2}\int(1+\rho)u\langle v,|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f_t|^2\rangle\,\mathrm{d}x+\int\partial^\alpha a_t u\cdot\partial^\alpha(b_t-u_t)\,\mathrm{d}x\\
&+\int(1+\rho)u\cdot\langle v\mathbf{P}\partial^\alpha f_t,\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f_t\rangle\,\mathrm{d}x+\int\rho\partial^\alpha a_t u\cdot\partial^\alpha b_t\,\mathrm{d}x\\
&+\sum_{1\leq\gamma\leq\alpha}C_{\alpha,\gamma}\Big(\iint\partial^\gamma\big((1+\rho)u\big)\cdot\frac{v}{2}\partial^{\alpha-\gamma}f_t\partial^\alpha f_t\,\mathrm{d}x\mathrm{d}v-\int\partial^{\alpha-\gamma}a_t\partial^\gamma u\cdot\partial^\alpha u_t\,\mathrm{d}x\Big).
\end{aligned}
$$

For the terms on the right hand side of the above equality, we have

$$
\begin{aligned}
&\frac{1}{2}\int(1+\rho)u\cdot\langle v,|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f_t|^2\rangle\,\mathrm{d}x\leq C(1+\|\nabla\rho\|_{H^1})\|\nabla u\|_{H^1}\|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f_t\|_\nu^2,\\
&\int(1+\rho)u\cdot\langle v\mathbf{P}\partial^\alpha f_t,\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f_t\rangle\,\mathrm{d}x\leq C\|(1+\rho)u\|_{L^\infty}\|\partial^\alpha f_t\|\|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f_t\|\\
&\qquad\qquad\qquad\leq\eta\|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f_t\|^2+C_\eta(1+\|\rho\|_{H^2}^2)\|\nabla u\|_{H^1}^2\|\partial^\alpha f_t\|^2,\\
&\int u\partial^\alpha a_t\cdot\partial^\alpha(b_t-u_t)\,\mathrm{d}x\leq C\|\nabla u\|_{H^1}\|\partial^\alpha a_t\|\|\partial^\alpha(u_t-b_t)\|\\
&\qquad\qquad\qquad\leq\eta\|\partial^\alpha(b_t-u_t)\|^2+C_\eta\|\nabla u\|_{H^1}^2\|f_t\|_{L_v^2(H_x^3)}^2,\\
&\int\rho\partial^\alpha a_t u\cdot\partial^\alpha b_t\,\mathrm{d}x\leq C\|\rho u\|_{L^\infty}\|\partial^\alpha a_t\|\|\partial^\alpha b_t\|\leq C(\|\nabla\rho\|_{H^1}^2+\|\nabla u\|_{H^1}^2)\|f_t\|_{L_v^2(H_x^3)}^2,\\
&\int\partial^\gamma u\partial^{\alpha-\gamma}a_t\cdot\partial^\alpha u_t\,\mathrm{d}x\leq C\|\nabla u\|_{H^2}\|\nabla a_t\|_{H^1}\|u_t\|_{H^3}
\end{aligned}
$$

$$\leq \epsilon\|\nabla a_t\|_{H^1}^2 + C_\epsilon\|\nabla u\|_{H^2}^2\|u_t\|_{H^3}^2,$$

$$\begin{aligned}
&\iint \partial^\gamma\big((1+\rho)u\big)\cdot\frac{v}{2}\partial^{\alpha-\gamma}f_t\partial^\alpha f_t\,\mathrm{d}x\mathrm{d}v\\
&=\int \partial^\gamma\big((1+\rho)u\big)\cdot\frac{v}{2}\Big(\partial^{\alpha-\gamma}\{\mathbf{I}-\mathbf{P}\}f_t+\partial^{\alpha-\gamma}\mathbf{P}f_t\Big)\partial^\alpha f_t\,\mathrm{d}x\mathrm{d}v\\
&\leq C\big(1+\|\rho\|_{H^3}\big)\|\nabla u\|_{H^2}\Big\{\sum_{|\alpha|\leq 2}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu+\|\nabla(a_t,b_t)\|_{H^1}\Big\}\|\partial^\alpha f_t\|\\
&\leq \eta\sum_{|\alpha|\leq 2}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu + C_\eta(1+\|\rho\|_{H^3}^2)\|\nabla u\|_{H^2}^2\|f_t\|_{L_v^2(H_x^3)}^2\\
&\quad+\epsilon\|\nabla(a_t,b_t)\|_{H^1}^2+C_\epsilon(1+\|\rho\|_{H^3}^2)\|\nabla u\|_{H^2}^2\|f_t\|_{L_v^2(H_x^3)}^2,
\end{aligned}$$

where ϵ,η are sufficiently small constant. Thus the third and forth terms have the following bound:

$$\begin{aligned}
&\iint \partial^\alpha\big((1+\rho)u\cdot\frac{v}{2}f_t\big)\partial^\alpha f_t\,\mathrm{d}x\mathrm{d}v-\int\partial^\alpha(ua_t)\cdot\partial^\alpha u_t\,\mathrm{d}x\\
&\leq C(1+\|\nabla\rho\|_{H^1})\|\nabla u\|_{H^1}\|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f_t\|_\nu^2+C\|\nabla(\rho,u)\|_{H^1}^2\|f_t\|_{L_v^2(H_x^3)}^2\\
&+\eta\Big\{\sum_{|\alpha|\leq 2}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu+\|\partial^\alpha(u_t-b_t)\|^2\Big\}+C_\eta(1+\|\rho\|_{H^3}^2)\|\nabla u\|_{H^2}^2\|f_t\|_{L_v^2(H_x^3)}^2\\
&+\epsilon\|\nabla(a_t,b_t)\|_{H^1}^2+C_\epsilon(1+\|\rho\|_{H^3}^2)\|\nabla u\|_{H^2}^2\|f_t\|_{L_v^2(H_x^3)}^2+C_\epsilon\|\nabla u\|_{H^2}^2\|u_t\|_{H^3}^2. \quad (3.20)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\iint[\partial^\alpha,(1+\rho)u\cdot\nabla_v]f_t\partial^\alpha f_t\,\mathrm{d}x\mathrm{d}v\\
&\quad\leq\eta\sum_{|\alpha|\leq 2}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu+C_\eta(1+\|\rho\|_{H^3}^2)\|\nabla u\|_{H^2}^2\|f_t\|_{L_v^2(H_x^3)}^2\\
&\quad\quad+\epsilon\|\nabla(a_t,b_t)\|_{H^1}^2+C_\epsilon(1+\|\rho\|_{H^3}^2)\|\nabla u\|_{H^2}^2\|f_t\|_{L_v^2(H_x^3)}^2. \quad (3.21)
\end{aligned}$$

Next, we deal with $\iint\partial^\alpha(\rho_t\mathcal{L}f)\partial^\alpha f_t\,\mathrm{d}x\mathrm{d}v$. We have

$$\begin{aligned}
&\iint\partial^\alpha(\rho_t\mathcal{L}f)\partial_x^\alpha f_t\,\mathrm{d}x\mathrm{d}v\\
&\quad=-\iint\partial^\alpha\Big(\rho_t\big(\nabla_v+\frac{v}{2}\big)f\Big)\cdot\partial^\alpha b_t\sqrt{M}\,\mathrm{d}x\mathrm{d}v\\
&\quad\quad-\iint\partial^\alpha\Big(\rho_t\big(\nabla_v+\frac{v}{2}\big)f\Big)\cdot\Big(\nabla_v+\frac{v}{2}\Big)\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}f_t\,\mathrm{d}x\mathrm{d}v\\
&\quad\leq C\|\nabla\rho_t\|_{H^2}\Big\|\Big(\nabla_v+\frac{v}{2}\Big)\nabla_x f\Big\|_{L_v^2(H_x^2)}\Big(\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu+\|\partial^\alpha b_t\|\Big)\\
&\quad\leq\eta\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu^2+C_\eta\Big\{\sum_{|\alpha|\leq 3}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\|\nabla(a,b)\|_{H^2}^2\Big\}\|\rho_t\|_{H^3}^2\\
&\quad\quad+\epsilon\|\nabla b_t\|_{H^2}^2+C_\epsilon\Big\{\sum_{|\alpha|\leq 3}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\|\nabla(a,b)\|_{H^2}^2\Big\}\|\rho_t\|_{H^3}^2. \quad (3.22)
\end{aligned}$$

Similarly, we have

$$-\iint\partial^\alpha(\rho\mathcal{L}f_t)\partial_x^\alpha f_t\,\mathrm{d}x\mathrm{d}v$$

$$
\begin{aligned}
&= \iint \partial^\alpha \Big(\rho(\nabla_v + \frac{v}{2}) f_t\Big) \cdot \Big(\nabla_v + \frac{v}{2}\Big)\partial_x^\alpha f_t \,\mathrm{d}x\mathrm{d}v \\
&\le C\|\nabla\rho\|_{H^2}\Big\{\sum_{|\alpha|\le 3}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu^2 + \|\nabla(a_t,b_t)\|_{H^2}^2\Big\}. \qquad (3.23)
\end{aligned}
$$

For the sake of brevity, we only give the bound of the remaining terms on the right hand side of (3.18) as follows:

$$
\int \partial^\alpha(\rho_t u + \rho u_t)\cdot\partial^\alpha b_t \,\mathrm{d}x \le \epsilon\|\nabla b_t\|_{H^2}^2 + C_\epsilon\|\nabla(\rho,u)\|_{H^2}^2\|(\rho_t,u_t)\|_{H^3}^2, \qquad (3.24)
$$

$$
\begin{aligned}
&\int \partial^\alpha u_t \cdot \nabla\frac{p'(1+\rho)}{1+\rho}\partial^\alpha\rho_t \,\mathrm{d}x + \int [\partial^\alpha, \rho\nabla\cdot]u_t\frac{p'(1+\rho)}{(1+\rho)^2}\partial^\alpha\rho_t\,\mathrm{d}x \\
&\qquad \le \eta\|\nabla u_t\|_{H^2}^2 + C_\eta\|\nabla\rho\|_{H^2}^2\|\rho_t\|_{H^3}^2, \qquad (3.25)
\end{aligned}
$$

$$
\begin{aligned}
&\int \frac{p'(1+\rho)}{(1+\rho)^2}u\cdot\nabla\partial^\alpha\rho_t\partial^\alpha\rho_t\,\mathrm{d}x \\
&\qquad \le C\|\nabla\rho\|_{H^2}\|\nabla u\|_{H^1}\|\partial^\alpha\rho_t\|^2 + C\|u\|_{H^3}\|\rho_t\|_{H^3}^2, \qquad (3.26)
\end{aligned}
$$

$$
\begin{aligned}
&\int [-\partial^\alpha, u\cdot\nabla]\rho_t\frac{p'(1+\rho)}{(1+\rho)^2}\partial^\alpha\rho_t\,\mathrm{d}x - \int \partial^\alpha\big(\rho_t\,\mathrm{div}\,u + u_t\cdot\nabla\rho\big)\frac{p'(1+\rho)}{(1+\rho)^2}\partial^\alpha\rho_t\,\mathrm{d}x \\
&\qquad \le \eta\|\nabla u_t\|_{H^2}^2 + C_\eta\|\nabla\rho\|_{H^3}^2\|\nabla\rho_t\|_{H^2}^2 + C\|u\|_{H^4}\|\nabla\rho_t\|_{H^2}^2, \qquad (3.27)
\end{aligned}
$$

$$
\begin{aligned}
&\int u\cdot\nabla\partial^\alpha u_t\cdot\partial^\alpha u_t\,\mathrm{d}x + \int \nabla\frac{1}{1+\rho}\cdot\nabla\partial^\alpha u_t\cdot\partial^\alpha u_t\,\mathrm{d}x \\
&\qquad + \sum_{1\le\gamma\le\alpha} C_{\alpha,\gamma}\int\partial^\gamma\Big(\frac{1}{1+\rho}\Big)\partial^{\alpha-\gamma}\Delta u_t\partial^\alpha u_t\,\mathrm{d}x \\
&\qquad \le \eta\|\nabla\partial^\alpha u_t\|^2 + C_\eta\big(\|\nabla(\rho,u)\|_{H^2}^2 + \|\nabla\rho\|_{H^2}^3\big)\|u_t\|_{H^3}^2, \qquad (3.28)
\end{aligned}
$$

$$
\begin{aligned}
&\int -\partial^\alpha(u_t\cdot\nabla u)\partial^\alpha\,\mathrm{d}x + \int [-\partial^\alpha, u\cdot\nabla]u_t\partial^\alpha u_t\,\mathrm{d}x \\
&\qquad + \int\Big[-\partial^\alpha, \frac{p'(1+\rho)}{1+\rho}\nabla\Big]\rho_t\partial^\alpha u_t\,\mathrm{d}x \\
&\qquad \le \eta\|\nabla u_t\|_{H^2}^2 + C_\eta\big(\|(\nabla\rho,\nabla u)\|_{H^3}^2 + \|\nabla\rho\|_{H^2}^3\big)\|(\rho_t,u_t)\|_{H^3}^2, \qquad (3.29)
\end{aligned}
$$

$$
\int\partial_t\Big(\frac{p'(1+\rho)}{(1+\rho)^2}\Big)|\partial^\alpha\rho_t|^2\,\mathrm{d}x \le C(1+\|\rho\|_{H^3})\|u\|_{H^3}\|\nabla\rho_t\|_{H^2}^2, \qquad (3.30)
$$

$$
\begin{aligned}
&\int\partial^\alpha\Big(\frac{1}{(1+\rho)^2}\rho_t\Delta u\Big)\cdot\partial^\alpha u_t\,\mathrm{d}x \\
&\qquad \le C(1+\|\rho\|_{H^3}^4)\|\nabla\rho\|_{H^2}^2\|\rho_t\|_{H^3}^2 + C\|\nabla u\|_{H^4}^2\|u_t\|_{H^3}^2, \qquad (3.31)
\end{aligned}
$$

$$
\int\partial^\alpha\big((1+\rho)^{\gamma-3}\rho_t\nabla\rho\big)\cdot\partial^\alpha u_t\,\mathrm{d}x \le C(1+\|\rho\|_{H^3}^2)\|\nabla\rho\|_{H^3}^2\|(\rho_t,u_t)\|_{H^3}^2, \qquad (3.32)
$$

$$
\int\partial^\alpha(au_t)\cdot\partial^\alpha u_t\,\mathrm{d}x \le \eta\|\nabla u_t\|_{H^2}^2 + C_\eta\|\nabla a\|_{H^2}^2\|u_t\|_{H^3}^2. \qquad (3.33)
$$

Finally, plugging the above estimates (3.19)-(3.33) into (3.18), using(1.12), and choosing η sufficiently small, we obtain (3.17). □

Remark 3.1. Combining Proposition 3.1 and Proposition 3.2, for the solution (f,ρ,u) obtained in Theorem 1.1, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\sum_{|\alpha|\le 3}\|\partial^\alpha f_t\|^2+\sum_{|\alpha|\le 3}\|\partial^\alpha u_t\|^2+\|\rho_t\|^2+\sum_{1\le|\alpha|\le 3}\Big\|\frac{\sqrt{p'(1+\rho)}}{1+\rho}\partial^\alpha\rho_t\Big\|^2\Big\}\\
&\quad+\lambda\sum_{|\alpha|\le 3}\big\{\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu^2+\|\nabla\partial^\alpha u_t\|^2+\|\partial^\alpha(u_t-b_t)\|^2\big\}\\
&\le\epsilon\big\{\|\nabla a_t\|_{H^1}^2+\|\nabla b_t\|_{H^2}^2+\|\nabla\rho_t\|^2\big\}+C_\epsilon\big(1+\|\rho\|_{H^3}^2\big)\|\nabla u\|_{H^2}^2\|f_t\|_{L_v^2(H_x^3)}^2\\
&\quad+C_\epsilon\Big\{\sum_{|\alpha|\le 3}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\|\nabla_x(a,b,\rho,u)\|_{H^2}^2+\|u-b\|^2\Big\}\|(\rho_t,u_t)\|_{H^3}^2\\
&\quad+C\|\rho\|_{H^3}\|\nabla(a_t,b_t)\|_{H^2}^2+C\big(1+\|\rho\|_{H^3}\big)\|u\|_{H^4}\|\nabla\rho_t\|_{H^2}^2\\
&\quad+C\big((1+\|\rho\|_{H^2})\|u\|_{H^2}+\|\rho\|_{H^3}\big)\sum_{1\le|\alpha|\le 3}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu^2\\
&\quad+C(1+\|\rho\|_{H^4}^4+\|u\|_{H^2}^2)\Big\{\sum_{|\alpha|\le 3}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\|\nabla(a,b)\|_{H^2}^2\\
&\quad+\|\nabla\rho\|_{H^3}^2+\|\nabla u\|_{H^4}^2\Big\}\big(\|f_t\|_{L_v^2(H_x^3)}^2+\|\rho_t,u_t\|_{H^3}^2\big).
\end{aligned}\tag{3.34}
$$

Next, we need to estimate $\|\nabla(a_t,b_t)\|_{H^2}^2$. From (4.12)-(4.14), we deduce that (a_t,b_t) satisfy the following system

$$\partial_t a_t+\operatorname{div} b_t=0,\tag{3.35}$$

$$
\begin{aligned}
&\partial_t b_{t,i}+\partial_i a_t+\sum_j\partial_j\Gamma_{i,j}(\{\mathbf{I}-\mathbf{P}\}f_t)\\
&\qquad=-(1+\rho)b_{t,i}-\rho_t b_i+\rho_t u_i(1+a)+(1+\rho)u_{t,i}(1+a)+(1+\rho)u_i a_t,
\end{aligned}\tag{3.36}
$$

$$
\begin{aligned}
&\partial_i b_{t,j}+\partial_j b_{t,i}-(1+\rho)(u_{t,i}b_j+u_{t,j}b_i+u_i b_{t,j}+u_j b_{t,i})-\rho_t(u_i b_j+u_j b_i)\\
&\qquad=-\partial_t\Gamma(\{\mathbf{I}-\mathbf{P}\}f_t)+\Gamma_{i,j}(l_t+r_t+s_t),
\end{aligned}\tag{3.37}
$$

where

$$
\begin{aligned}
&l_t:=-v\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}f_t+L\{\mathbf{I}-\mathbf{P}\}f_t,\\
&r_t:=\partial_t\Big(-u\cdot\nabla_v\{\mathbf{I}-\mathbf{P}\}f+\frac{1}{2}u\cdot v\{\mathbf{I}-\mathbf{P}\}f\Big),\\
&s_t:=\partial_t\Big(\rho M^{-\frac{1}{2}}\nabla_v\cdot\Big(\frac{v}{2}\sqrt{M}\{\mathbf{I}-\mathbf{P}\}f+\sqrt{M}\{\mathbf{I}-\mathbf{P}\}f-u\sqrt{M}\{\mathbf{I}-\mathbf{P}\}f\Big)\Big).
\end{aligned}
$$

Define the following functional

$$
\begin{aligned}
\bar{\mathcal{E}}_0(f_t(t)):=&\sum_{|\alpha|\le 2}\sum_{i,j}\int_{\mathbb{R}^3}\partial_x^\alpha(\partial_{x_j}b_{t,i}+\partial_{x_i}b_{t,j})\partial_x^\alpha\Gamma_{i,j}(\{\mathbf{I}-\mathbf{P}\}f_t)\,\mathrm{d}x\\
&-\sum_{|\alpha|\le 2}\int_{\mathbb{R}^3}\partial_x^\alpha a_t\partial_x^\alpha\nabla_x\cdot b_t\,\mathrm{d}x.
\end{aligned}\tag{3.38}
$$

Similarly to the proofs of Proposition 4.3 and Proposition 4.4, we have the following two results:

Proposition 3.3. *Assume that (f,ρ,u) is the solution obtained in Theorem 1.1. Then, we have*

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mathcal{E}}_0(f_t(t))+\frac{3}{2}\|\nabla b_t\|_{H^2}^2+\frac{1}{2}\|\nabla\cdot b_t\|_{H^2}^2+\frac{1}{4}\|\nabla a_t\|_{H^2}^2\\
\le\, & C\big(\|u_t-b_t\|_{H^2}^2+\|\{\mathbf{I}-\mathbf{P}\}f_t\|_{L^2_x(H^3_x)}^2\big)+C\big(1+\|\rho\|_{H^3}^2+\|u\|_{H^3}^2\big)\\
&\times\big(\|\nabla(a,b,u)\|_{H^2}^2+\|\{\mathbf{I}-\mathbf{P}\}f\|_{L^2_x(H^3_x)}^2+\|u-b\|_{H^2}^2\big)\big(\|f_t\|_{L^2_x(H^3_x)}^2+\|(\rho_t,u_t)\|_{H^3}^2\big)\\
&+C\big(\|(\rho,u)\|_{H^3}^2+\|\rho\|_{H^3}^2\|u\|_{H^3}^2\big)\big(\|u_t-b_t\|_{H^2}^2+\|\{\mathbf{I}-\mathbf{P}\}f_t\|_{L^2_x(H^3_x)}^2\big).
\end{aligned}\tag{3.39}
$$

Proposition 3.4. *Assume that (f,ρ,u) is the solution obtained in Theorem 1.1. Then, we have*

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\partial^\alpha u_t\partial^\alpha\nabla\rho_t\,\mathrm{d}x+\frac{1}{2}p'(1)\|\nabla\partial^\alpha\rho_t\|^2\\
\le\, & C\big(\|\nabla u_t\|_{H^3}^2+\|u_t-b_t\|_{H^2}^2\big)+C\big(1+\|\rho\|_{H^3}^4\big)\\
&\big(\|\nabla(a,\rho)\|_{H^2}^2+\|\nabla u\|_{H^3}^2\big)\big(\|f_t\|_{L^2_x(H^3_x)}^2+\|(\rho_t,u_t)\|_{H^3}^2\big)\\
&+C\|(\rho,u)\|_{H^3}\|\nabla(\rho_t,u_t)\|_{H^2}^2.
\end{aligned}\tag{3.40}
$$

Now, we define a temporal energy functional $\bar{\mathcal{E}}_1(t)$ and the corresponding dissipation rate $\bar{\mathcal{D}}_1(t)$ by

$$
\begin{aligned}
\bar{\mathcal{E}}_1(t):=\,&\|f_t\|^2+\|\rho_t\|^2+\|u_t\|^2+\sum_{1\le|\alpha|\le3}\bigg\{\|\partial^\alpha f_t\|^2+\bigg\|\frac{\sqrt{p'(1+\rho)}}{1+\rho}\partial^\alpha\rho_t\bigg\|^2\\
&+\|\partial^\alpha u_t\|^2\bigg\}+\kappa_1\bar{\mathcal{E}}_0(t)+\kappa_2\sum_{|\alpha|\le2}\int_{\mathbb{R}^3}\partial^\alpha u_t\cdot\partial^\alpha\nabla\rho_t\,\mathrm{d}x,\\
\bar{\mathcal{D}}_1(t):=\,&\|\nabla(a_t,b_t,\rho_t,u_t)\|_{H^2}^2+\|b_t-u_t\|_{H^3}^2+\sum_{|\alpha|\le3}\Big\{\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu^2+\|\partial^\alpha\nabla u_t\|^2\Big\},
\end{aligned}
$$

where $\kappa_1,\kappa_2>0$ are small constants to be chosen later. Obviously, we have

$$
\bar{\mathcal{E}}_1(t)\sim\|f_t(t)\|_{L^2_v(H^3_x)}^2+\|(\rho_t,u_t)(t)\|_{H^3}^2.
$$

According to (3.34), (3.39), (3.40), and by choosing κ_1, κ_2 and ϵ sufficiently small, we finally obtain that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mathcal{E}}_1(t)+\lambda\sum_{|\alpha|\le3}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu^2\\
&+\lambda\Big(\|\nabla(a_t,b_t,\rho_t,u_t)\|_{H^2}^2+\|u_t-b_t\|_{H^3}^2+\|\nabla u_t\|_{H^3}^2\Big)\\
\le\, & C\Big\{\|\rho\|_{H^3}+\|\rho\|_{H^3}^2+\|u\|_{H^4}+\|u\|_{H^3}^2+\|\rho\|_{H^3}\|u\|_{H^4}+\|\rho\|_{H^3}^2\|u\|_{H^3}^2\Big\}\\
&\times\Big\{\sum_{|\alpha|\le3}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu^2+\|\nabla(a_t,b_t,\rho_t,u_t)\|_{H^2}^2+\|u_t-b_t\|_{H^2}^2\Big\}\\
&+C(1+\|\rho\|_{H^4}^4+\|u\|_{H^3}^2)\Big\{\sum_{|\alpha|\le3}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\|\nabla(a,b)\|_{H^2}^2\\
&+\|\nabla\rho\|_{H^3}^2+\|\nabla u\|_{H^4}^2+\|u-b\|_{H^2}^2\Big\}\big(\|f_t\|_{L^2_v(H^3_x)}^2+\|(\rho_t,u_t)\|_{H^3}^2\big).
\end{aligned}
$$

For simplicity, by the definition of $\mathcal{E}_1(t), \mathcal{D}_1(t)$ in (1.14),(1.15), we can rewrite the above inequality as

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mathcal{E}}_1(t)+\lambda\bar{\mathcal{D}}_1(t)\le C\big(\mathcal{E}_1^{\frac{1}{2}}(t)+\mathcal{E}_1^2(t)\big)\bar{\mathcal{D}}_1(t)+C(1+\mathcal{E}_1^2(t))\mathcal{D}_1(t)\bar{\mathcal{E}}_1(t), \tag{3.41}$$

Next, we need to estimate the mixed space-velocity derivatives of f_t. Since $\|\partial_\beta^\alpha \mathbf{P} f_t\|\le C\|\partial^\alpha f_t\|$ for any α,β, we only estimate $\|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f_t\|$. From (4.23), we easily deduce that

$$\begin{aligned}
&\partial_t\{\mathbf{I}-\mathbf{P}\}f_t+v\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}f_t+u\cdot\nabla_v\{\mathbf{I}-\mathbf{P}\}f_t-\frac{1}{2}u\cdot v\{\mathbf{I}-\mathbf{P}\}f_t\\
=&\mathcal{L}\{\mathbf{I}-\mathbf{P}\}f_t+\mathbf{P}\big(v\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}f_t+u\cdot\nabla_v\{\mathbf{I}-\mathbf{P}\}f_t-\frac{1}{2}u\cdot v\{\mathbf{I}-\mathbf{P}\}f_t\big)\\
&-\{\mathbf{I}-\mathbf{P}\}\Big(v\cdot\nabla_x\mathbf{P}f_t+u\cdot\nabla_v\mathbf{P}f_t-\frac{1}{2}u\cdot v\mathbf{P}f_t\Big)+\{\mathbf{I}-\mathbf{P}\}G_t,
\end{aligned}\tag{3.42}$$

where $\{\mathbf{I}-\mathbf{P}\}G_t$ is defined by

$$\begin{aligned}
&\{\mathbf{I}-\mathbf{P}\}G_t\\
=&\rho\mathcal{L}\{\mathbf{I}-\mathbf{P}\}f_t+\rho_t\mathcal{L}\{\mathbf{I}-\mathbf{P}\}f-\rho u\nabla_v\{\mathbf{I}-\mathbf{P}\}f_t+\frac{1}{2}\rho u\cdot v\{\mathbf{I}-\mathbf{P}\}f_t\\
&-(\rho_t u+(1+\rho)u_t)\cdot\Big(\nabla_v-\frac{v}{2}\Big)\{\mathbf{I}-\mathbf{P}\}f\\
&+\mathbf{P}\Big\{\rho u\cdot\Big(\nabla_v-\frac{v}{2}\Big)\{\mathbf{I}-\mathbf{P}\}f_t+(\rho_t u+(1+\rho)u_t)\cdot\Big(\nabla_v-\frac{v}{2}\Big)\{\mathbf{I}-\mathbf{P}\}f\Big\}\\
&-\{\mathbf{I}-\mathbf{P}\}\Big\{\rho u\cdot\Big(\nabla_v-\frac{v}{2}\Big)\mathbf{P}f_t+(\rho_t u+(1+\rho)u_t)\cdot\Big(\nabla_v-\frac{v}{2}\Big)\mathbf{P}f\Big\}.
\end{aligned}$$

Similarly to the proof of Proposition 4.5, we can prove that

Proposition 3.5. *Assume that (f,ρ,u) is the solution obtained in Theorem 1.1. Then we have*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\substack{|\beta|=k\\|\alpha|+|\beta|\le3}}\|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f_t\|^2+\lambda\sum_{\substack{|\beta|=k\\|\alpha|+|\beta|\le3}}\|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu^2\\
\le&C\Big\{\sum_{|\alpha'|\le4-k}\|\partial^{\alpha'}\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu^2+\|\nabla b_t\|_{H^{3-k}}^2\Big\}+C\chi_{2\le k\le4}\sum_{\substack{1\le|\beta'|\le k-1\\|\alpha'|+|\beta'|\le3}}\|\partial_{\beta'}^{\alpha'}\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu^2\\
&+C(1+\|\rho\|_{H^3}^2)\|\nabla u\|_{H^2}^2\Big\{\sum_{|\alpha'|\le3-k}\|\partial^{\alpha'}\{\mathbf{I}-\mathbf{P}\}f_t\|^2+\sum_{\substack{1\le|\beta'|\le3\\|\alpha'|+|\beta'|\le3}}\|\partial_{\beta'}^{\alpha'}\{\mathbf{I}-\mathbf{P}\}f_t\|^2\Big\}\\
&+C(1+\|\rho\|_{H^3}^2+\|u\|_{H^3}^2)\Big\{\sum_{|\alpha'|+|\beta'|\le3}\|\partial_{\beta'}^{\alpha'}\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\|\nabla b\|_{H^2}^2\Big\}\|(\rho_t,u_t)\|_{H^3}^2\\
&+C(\|\rho\|_{H^3}^2+\|u\|_{H^3}^2+\|\rho\|_{H^3}^2\|u\|_{H^3}^2)\Big\{\sum_{|\alpha'|\le3-k}\|\partial^{\alpha'}\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu^2\\
&+\sum_{\substack{1\le|\beta'|\le3\\|\alpha'|+|\beta'|\le3}}\|\partial_{\beta'}^{\alpha'}\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu^2+\|\nabla b_t\|_{H^2}^2\Big\}.
\end{aligned}\tag{3.43}$$

Multiplying (3.43) by suitable constants R_k and taking summation over k, we obtain the following inequality:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{1\le k\le 3} R_k \sum_{\substack{|\beta|=k\\|\alpha|+|\beta|\le 3}} \|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f_t\|^2+\lambda\sum_{\substack{1\le|\beta|\le 3\\|\alpha|+|\beta|\le 3}}\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu^2\\
\le{}& C\Big\{\sum_{|\alpha|\le 3}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu^2+\|\nabla b_t\|_{H^2}^2\Big\}\\
&+C(1+\|\rho\|_{H^3}^2)\|\nabla u\|_{H^2}^2\Big\{\sum_{|\alpha|\le 2}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f_t\|^2+\sum_{\substack{1\le|\beta|\le 3\\|\alpha|+|\beta|\le 3}}\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f_t\|^2\Big\}\\
&+C(1+\|\rho\|_{H^3}^2+\|u\|_{H^3}^2)\Big\{\sum_{|\alpha'|+|\beta'|\le 3}\|\partial^{\alpha'}_{\beta'}\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\|\nabla b\|_{H^2}^2\Big\}\|(\rho_t,u_t)\|_{H^3}^2\\
&+C(\|\rho\|_{H^3}^2+\|u\|_{H^3}^2+\|\rho\|_{H^3}^2\|u\|_{H^3}^2)\Big\{\sum_{|\alpha|\le 3}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu^2\\
&+\sum_{\substack{1\le|\beta|\le 3\\|\alpha|+|\beta|\le 3}}\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu^2+\|\nabla b_t\|_{H^2}^2\Big\}. \qquad (3.44)
\end{aligned}
$$

Now, we define $\bar{\mathcal{E}}_2(t)$ and $\bar{\mathcal{D}}_2(t)$ as

$$
\begin{aligned}
\bar{\mathcal{E}}_2(t)&:=\sum_{1\le k\le 3}R_k\sum_{\substack{|\beta|=k\\|\alpha|+|\beta|\le 3}}\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f_t\|^2,\\
\bar{\mathcal{D}}_2(t)&:=\sum_{\substack{1\le|\beta|\le 3\\|\alpha|+|\beta|\le 3}}\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f_t\|_\nu^2.
\end{aligned}
$$

Therefore, according to the definition of $\mathcal{D}_2(t)$ in (1.17), (3.44) can be rewrited as

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mathcal{E}}_2(t)+\lambda\bar{\mathcal{D}}_2(t)\le{}& C\bar{\mathcal{D}}_1(t)+C\big(\mathcal{E}_1(t)+\mathcal{E}_1^2(t)\big)\big(\bar{\mathcal{D}}_1(t)+\bar{\mathcal{D}}_2(t)\big)\\
&+C\big(1+\mathcal{E}_1(t)\big)\big(\mathcal{D}_1(t)+\mathcal{D}_2(t)\big)\big(\bar{\mathcal{E}}_1(t)+\bar{\mathcal{E}}_2(t)\big). \qquad (3.45)
\end{aligned}
$$

Thus, we define a total energy functional $\bar{\mathcal{E}}(t)$ and the corresponding dissipation rate $\bar{\mathcal{D}}(t)$ by

$$
\begin{aligned}
\bar{\mathcal{E}}(t)&:=\bar{\mathcal{E}}_1(t)+\kappa_3\bar{\mathcal{E}}_2(t),\\
\bar{\mathcal{D}}(t)&:=\bar{\mathcal{D}}_1(t)+\kappa_3\bar{\mathcal{D}}_2(t),
\end{aligned}
$$

where $\kappa_3>0$ is a small constant to be chosen later.

With the aid of (1.18),(1.19),(3.41) and (3.45), we conclude that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mathcal{E}}(t)+\lambda\bar{\mathcal{D}}\le C\big(\mathcal{E}^{\frac12}(t)+\mathcal{E}^2(t)\big)\bar{\mathcal{D}}+C\big(1+\mathcal{E}^2\big)\mathcal{D}\bar{\mathcal{E}}. \qquad (3.46)
$$

By the uniform a priori estimates obtained in Section 4, we deduce that $\mathcal{E}(t)$ is sufficiently small. Therefore, we obtain that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mathcal{E}}(t)+\lambda\bar{\mathcal{D}}\le C(1+\mathcal{E}^2)\mathcal{D}\bar{\mathcal{E}}. \qquad (3.47)
$$

Applying Grownwall's inequality to (3.47), we obtain

$$\sup_{0\le t<\infty}\bar{\mathcal{E}}(t)+\lambda\int_0^{+\infty}\bar{\mathcal{D}}\mathrm{d}t\le C\bar{\mathcal{E}}(0)<+\infty.$$

Meanwhile, according to equations (1.7)-(1.10) and the uniform a priori estimates obtained above, we deduce that,

$$\int_0^{+\infty}(\|f_t\|^2+\|\rho_t\|^2+\|u_t\|^2)\,\mathrm{d}t<+\infty.$$

Thus,

$$\int_0^{+\infty}(\|f_t\|^2_{H^3_{x,v}}+\|\rho_t\|^2_{H^3}+\|u_t\|^2_{H^3})\,\mathrm{d}t<+\infty.$$

On the other hand, we can apply Lemma 3.1 to obtain

$$\|f_t\|^2_{H^3_{x,v}}+\|\rho_t\|^2_{H^3}+\|u_t\|^2_{H^3}\le\frac{C}{1+t}. \tag{3.48}$$

Now, we define the following functionals

$$\begin{aligned}
\tilde{\mathcal{E}}_0(f_t(t)):=&\sum_{|\alpha|\le2}\sum_{i,j}\int\partial_x^\alpha(\partial_{x_j}b_i+\partial_{x_i}b_j)\partial_x^\alpha\Gamma_{i,j}(\{\mathbf{I}-\mathbf{P}\}f)\,\mathrm{d}x\\
&-\sum_{|\alpha|\le2}\int\partial_x^\alpha a\partial_x^\alpha\nabla_x\cdot b\,\mathrm{d}x,\\
\tilde{\mathcal{E}}_1(t):=&\|f\|^2+\|\rho\|^2+\|u\|^2+\sum_{1\le|\alpha|\le3}\left\{\|\partial^\alpha f\|^2+\left\|\frac{\sqrt{p'(1+\rho)}}{1+\rho}\partial^\alpha\rho\right\|^2+\|\partial^\alpha u\|^2\right\}\\
&+\tau_1\tilde{\mathcal{E}}_0(t)+\tau_2\sum_{|\alpha|\le2}\int\partial^\alpha u\cdot\partial^\alpha\nabla\rho\,\mathrm{d}x,\\
\tilde{\mathcal{D}}_1(t):=&\|\nabla(a,b,\rho,u)\|^2_{H^2}+\|b-u\|^2_{H^3}+\sum_{|\alpha|\le3}\left\{\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2_\nu+\|\partial^\alpha\nabla u\|^2\right\},\\
\tilde{\mathcal{E}}_2(t):=&\sum_{1\le k\le3}C_k\sum_{\substack{|\beta|:=k\\|\alpha|+|\beta|\le3}}\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2,\\
\tilde{\mathcal{D}}_2(t):=&\sum_{\substack{1\le|\beta|\le3\\|\alpha|+|\beta|\le3}}\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2_\nu,\\
\tilde{\mathcal{E}}(t):=&\tilde{\mathcal{E}}_1(t)+\tau_3\tilde{\mathcal{E}}_2(t),\\
\tilde{\mathcal{D}}(t):=&\tilde{\mathcal{D}}_1(t)+\tau_3\tilde{\mathcal{D}}_2(t).
\end{aligned}$$

According to the results obtained in Section 4, we finally obtain that

$$\begin{aligned}
&\sup_{x\in\mathbb{R}^3}\Big\{\sum_{|\alpha|+|\beta|\le1}\|\partial^\alpha_\beta f\|^2_{L^2_v}\Big\}+\|\rho\|^2_{L^\infty}+\|u\|^2_{L^\infty}\\
\le&\sum_{|\alpha|+|\beta|\le2}\|\nabla\partial^\alpha_\beta f\|^2+\|\nabla\rho\|^2_{H^2}+\|\nabla u\|^2_{H^2}\le\tilde{\mathcal{D}}(t)\le-\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}(t)\\
\le&C(\|f\|_{H^3_{x,v}}+\|(\rho,u)\|_{H^3})(\|f_t\|_{H^3_{x,v}}+\|(\rho_t,u_t)\|_{H^3})\le C(1+t)^{-\frac{1}{2}}.
\end{aligned} \tag{3.49}$$

This completes the proof of Theorem 1.1.

3.2. **The case of periodic domain.** In this subsection we study the large time behavior of classical solutions when Ω is a spatial periodic domain $\mathbb{T}^3$. We shall use the uniform a priori estimates obtained in Section 4 below. It follows from (4.33) together with (4.35) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t)+\lambda\mathcal{D}_{\mathbb{T}}(t)\le C(\mathcal{E}^{\frac{1}{2}}(t)+\mathcal{E}^2(t))\mathcal{D}_{\mathbb{T}}(t).$$

Using the fact that $\mathcal{E}(t)$ is small enough and uniformly in time, and $\mathcal{E}(t)\le C\mathcal{D}_{\mathbb{T}}(t)$, we then obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t)+\lambda\mathcal{E}(t)\le 0$$

for all $t\ge 0$. This gives the desired exponential decay by applying Gronwall's inequality, and hence completes the proof of Theorem 1.2.

4. A priori estimates of the classical solutions

In this section, we shall establish the uniform-in-time a priori estimates in the spatial domain $\Omega=\mathbb{R}^3$ or $\mathbb{T}^3$ which have been used in Sections 2 and 3. We need the following two assumptions:

(1) (f,ρ,u) is the smooth solution to the Cauchy problem (1.7)-(1.10) for $0<t<T$ with a fixed $T>0$;
(2) (f,ρ,u) satisfies

$$\sup_{0<t<T}\{\|f(t)\|_{H^4_{x,v}}+\|(\rho,u)(t)\|_{H^4_x}\}\le\delta, \tag{4.1}$$

where $0<\delta<1$ is a sufficiently small generic constant.

First, we introduce a lemma which is useful in the subsequent estimates:

Lemma 4.1 (see [12]). *There exists a positive constant C, such that for any $f,g\in H^4(\Omega)$ and any multi-index γ with $1\le|\gamma|\le 4$,*

$$\|f\|_{L^\infty(\Omega)}\le C\|\nabla_x f\|_{L^2(\Omega)}^{1/2}\|\nabla_x^2 f\|_{L^2(\Omega)}^{1/2}, \tag{4.2}$$

$$\|fg\|_{H^2(\Omega)}\le C\|f\|_{H^2(\Omega)}\|\nabla_x g\|_{H^2(\Omega)}, \tag{4.3}$$

$$\|\partial_x^\gamma(fg)\|_{L^2(\Omega)}\le C\|\nabla_x f\|_{H^3(\Omega)}\|\nabla_x g\|_{H^3(\Omega)}. \tag{4.4}$$

4.1. **The case of the whole space.** In this subsection we deal with the uniform-in-time a priori estimates in the whole space $\Omega=\mathbb{R}^3$.

Proposition 4.1. *For smooth solutions of the problem* (1.7)-(1.10)*, we have*

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(f,\rho,u)(t)\|^2+\lambda(\|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2+\|b-u\|^2+\|\nabla u\|^2)\\ \le& C(\|(\rho,u)\|_{H^2}+\|\rho\|_{H^2}\|u\|_{H^2})(\|\nabla_x(a,b,\rho,u)\|^2+\|u-b\|^2+\|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2)\end{aligned} \tag{4.5}$$

for all $0\le t<T$.

Proof. Multiplying (1.7)-(1.9) by f, ρ, and u respectively and then taking integration and summation, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|f\|^2+\|\rho\|^2+\|u\|^2)+\int\langle -\mathcal{L}\{\mathbf{I}-\mathbf{P}\}f,f\rangle\,\mathrm{d}x+\int\frac{|\nabla u|^2}{1+\rho}\,\mathrm{d}x+\|b-u\|^2\\
=&\int u\Big\langle\frac{1}{2}vf,f\Big\rangle\,\mathrm{d}x-\int a|u|^2\,\mathrm{d}x-\int\rho\mathrm{div}\,u\,\mathrm{d}x-\int p'(1)\nabla\rho\cdot u\,\mathrm{d}x\\
&-\int(u\cdot\nabla u)\cdot u\,\mathrm{d}x-\int\nabla\frac{1}{1+\rho}\nabla u\cdot u\,\mathrm{d}x-\int\Big(\frac{p'(1+\rho)}{1+\rho}-p'(1)\Big)\nabla\rho\cdot u\,\mathrm{d}x\\
&-\frac{1}{2}\int\rho^2\mathrm{div}\,u\,\mathrm{d}x+\iint\rho\Big(\mathcal{L}f-u\cdot\nabla_v f+\frac{1}{2}u\cdot vf+u\cdot v\sqrt{M}\Big)f\,\mathrm{d}x\mathrm{d}v. \qquad (4.6)
\end{aligned}$$

By (1.12), we have

$$\langle -\mathcal{L}\{\mathbf{I}-\mathbf{P}\}f,f\rangle\ge\lambda_0|\{\mathbf{I}-\mathbf{P}\}f|_\nu^2.$$

Thus, we only need to estimate the terms on the right hand side of the equality (4.6). For the first two terms, by taking the same computation as that in [12], we get

$$\begin{aligned}
&\int u\Big\langle\frac{1}{2}vf,f\Big\rangle\,\mathrm{d}x-\int a|u|^2\,\mathrm{d}x\\
&\quad\le C(\|\nabla_v u\|_{H^1}+\|u\|_{H^1})\|\{\mathbf{I}-\mathbf{P}f\}\|_\nu^2+C\|u\|_{H^1}(\|\nabla_x(a,b)\|^2+\|u-b\|^2).
\end{aligned}$$

Without loss of generality, we can assume that $p'(1)=1$, then, one has

$$\int\rho\ \mathrm{div}\,u\,\mathrm{d}x+\int p'(1)\nabla\rho\cdot u\,\mathrm{d}x=0.$$

For the next four terms, using Hölder's, Sobolev's inequalities and Lemma 4.1, we have

$$\begin{aligned}
\int(u\cdot\nabla u)\cdot u\,\mathrm{d}x&\le C\|u\|_{L^3}\|\nabla_x u\|_{L^2}\|u\|_{L^6}\le C\|u\|_{H^1}\|\nabla u\|_{L^2}^2,\\
\int\nabla\frac{1}{1+\rho}\nabla u\cdot u\,\mathrm{d}x&\le C\|u\|_{H^2}(\|\nabla\rho\|^2+\|\nabla u\|^2),\\
\int\Big(\frac{p'(1+\rho)}{1+\rho}-p'(1)\Big)\nabla\rho\cdot u\,\mathrm{d}x&\le C\|\rho\|_{L^3}\|\nabla_x\rho\|_{L^2}\|u\|_{L^6}\\
&\le C\|\rho\|_{H^1}(\|\nabla\rho\|^2+\|\nabla u\|^2),\\
\frac{1}{2}\int\rho^2\ \mathrm{div}u\,\mathrm{d}x&\le C\|\rho\|_{L^3}\|\nabla_x u\|_{L^2}\|\rho\|_{L^6}\\
&\le C\|\rho\|_{H^1}(\|\nabla\rho\|^2+\|\nabla u\|^2).
\end{aligned}$$

Here, we have used the facts that $\|u\|_{L^6}\le C\|\nabla u\|_{L^2}$ and $\|u\|_{L^3}\le C\|u\|_{H^1}$.

Using the macro-micro decomposition (1.11), we rewrite the last term as

$$\begin{aligned}
&\iint\rho\Big(\mathcal{L}f-u\cdot\nabla_v f+\frac{1}{2}u\cdot vf+u\cdot v\sqrt{M}\Big)f\,\mathrm{d}x\mathrm{d}v\\
&\quad=\int\rho\langle -\mathcal{L}\{\mathbf{I}-\mathbf{P}\}f,f\rangle\,\mathrm{d}x+\int\rho(u-b)\cdot b\,\mathrm{d}x+\frac{1}{2}\iint\rho u\cdot vf^2\,\mathrm{d}x\mathrm{d}v.
\end{aligned}$$

It is easy to see that

$$\int \rho(u-b)\cdot b\,\mathrm{d}x \le C\|\rho\|_{L^3}\|u-b\|_{L^2}\|b\|_{L^6} \le C\|\rho\|_{H^1}(\|\nabla b\|_{L^2}^2+\|u-b\|_{L^2}^2).$$

Noticing that

$$\Big\langle \frac{1}{2}v_i f, f\Big\rangle = ab_i + \langle v_i\mathbf{P}f, \{\mathbf{I}-\mathbf{P}\}f\rangle + \Big\langle \frac{1}{2}v_i, |\{\mathbf{I}-\mathbf{P}\}f|^2\Big\rangle, \quad (i=1,2,3),$$

the term $\frac{1}{2}\iint \rho u\cdot v f^2\,\mathrm{d}x\mathrm{d}v$ can be estimated as follows:

$$\begin{aligned}
\int \rho u\cdot ba\,\mathrm{d}x &\le C\|\rho\|_{L^3}\|u\|_{L^3}\|a\|_{L^6}\|b\|_{L^6}\\
&\le C\|\rho\|_{H^1}\|u\|_{H^1}(\|\nabla a\|^2+\|\nabla b\|^2),\\
\int \rho\langle v_i\mathbf{P}f, \{\mathbf{I}-\mathbf{P}\}f\rangle\,\mathrm{d}x &\le C\|\rho u\|_{L^3}\|(a,b)\|_{L^6}\|\{\mathbf{I}-\mathbf{P}\}f\|\\
&\le C\|\rho\|_{H^1}\|u\|_{H^1}(\|\nabla(a,b)\|^2+\|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2),\\
\int \rho\Big\langle\frac{1}{2}v_i, |\{\mathbf{I}-\mathbf{P}\}f|^2\Big\rangle\,\mathrm{d}x &\le C\|\rho u\|_{L^\infty}\|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2\\
&\le C\|\rho\|_{H^2}\|u\|_{H^2}\|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2.
\end{aligned}$$

Therefore, the last term is bounded by

$$\begin{aligned}
&\int \rho\langle -\mathcal{L}\{\mathbf{I}-\mathbf{P}\}f, f\rangle\,\mathrm{d}x\\
&\quad + C\big(\|\rho\|_{H^1}+\|\rho\|_{H^2}\|u\|_{H^2}\big)\big(\|\nabla(a,b)\|^2+\|u-b\|^2+\|\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2\big).
\end{aligned}$$

Plugging all the above estimates into (4.6) and using (4.1), we obtain (4.5). □

Proposition 4.2. *For smooth solutions of the problem* (1.7)-(1.10), *we have*

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{1\le|\alpha|\le 4}\Big\{\|\partial^\alpha f\|^2+\Big\|\frac{\sqrt{p'(1+\rho)}}{1+\rho}\partial^\alpha\rho\Big\|^2+\|\partial^\alpha u\|^2\Big\}\\
&\quad+\lambda\sum_{1\le|\alpha|\le 4}\big\{\|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|_\nu^2+\|\partial^\alpha(b-u)\|^2+\|\nabla\partial^\alpha u\|^2\big\}\\
&\le C\big\{\|u\|_{H^4}+\|\rho\|_{H^4}\|u\|_{H^4}(1+\|\rho\|_{H^4}^2)+\|\rho\|_{H^4}(1+\|\rho\|_{H^4}^3)\big\}\\
&\quad\times\Big\{\|\nabla_x(a,b,\rho,u)\|_{H^3}^2+\sum_{1\le|\alpha'|\le 4}\|\{\mathbf{I}-\mathbf{P}\}\partial^{\alpha'}f\|_\nu^2\Big\}
\end{aligned}\tag{4.7}$$

for all $0\le t<T$.

Proof. Applying differentiation $\partial^\alpha(1\le|\alpha|\le 4)$ to the system (1.7)-(1.9), we have

$$\begin{aligned}
&\partial_t(\partial^\alpha f)+v\cdot\nabla_x(\partial^\alpha f)+u\cdot\nabla_v(\partial^\alpha f)-\partial^\alpha u\cdot v\sqrt{M}-\mathcal{L}\partial^\alpha f\\
&\quad=\frac{1}{2}\partial^\alpha[(1+\rho)u\cdot vf]+[-\partial^\alpha, u\cdot\nabla_v]f+\partial^\alpha\{\rho(\mathcal{L}f-u\cdot\nabla_v f+u\cdot v\sqrt{M})\},
\end{aligned}\tag{4.8}$$

$$\partial_t(\partial^\alpha\rho)+u\cdot\nabla\partial^\alpha\rho+(1+\rho)\mathrm{div}\,\partial^\alpha u=[-\partial^\alpha, \rho\nabla_x\cdot]u+[-\partial^\alpha, u\cdot\nabla_x]\rho,\tag{4.9}$$

$$\partial_t(\partial^\alpha u)+u\cdot\nabla(\partial^\alpha u)+\frac{p'(1+\rho)}{1+\rho}\nabla\partial^\alpha\rho-\partial^\alpha\Big(\frac{1}{1+\rho}\Delta u\Big)-\partial^\alpha(b-u)$$

$$= [-\partial^\alpha, u\cdot\nabla_x]u + \Big[-\partial^\alpha, \frac{p'(1+\rho)}{1+\rho}\nabla_x\Big]\rho - \partial^\alpha(ua), \tag{4.10}$$

where $[A,B]$ denotes the commutator $AB-BA$ for two operators A and B. Now, multiplying (4.8)-(4.10) by $\partial^\alpha f$, $\frac{p'(1+\rho)}{(1+\rho)^2}\partial^\alpha\rho$, and $\partial^\alpha u$ respectively and then taking integration and summation, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\|\partial^\alpha f\|^2 + \Big\|\frac{\sqrt{p'(1+\rho)}}{1+\rho}\partial^\alpha\rho\Big\|^2 + \|\partial^\alpha u\|^2\Big\}\\
&\quad + \int\langle -\mathcal{L}\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f, \partial^\alpha f\rangle\,\mathrm{d}x + \int\frac{1}{1+\rho}|\nabla(\partial^\alpha u)|^2\,\mathrm{d}x + \|\partial^\alpha(b-u)\|^2\\
&= \int\langle[-\partial^\alpha, u\cdot\nabla_v]f, \partial^\alpha f\rangle\,\mathrm{d}x + \int\frac{1}{2}\langle\partial^\alpha[(1+\rho)u\cdot vf], \partial^\alpha f\rangle\,\mathrm{d}x\\
&\quad + \int[-\partial^\alpha, \rho\nabla_x\cdot]u\frac{p'(1+\rho)}{(1+\rho)^2}\partial^\alpha\rho\,\mathrm{d}x + \int[-\partial^\alpha, u\cdot\nabla_x]\rho\frac{p'(1+\rho)}{(1+\rho)^2}\partial^\alpha\rho\,\mathrm{d}x\\
&\quad + \int[-\partial^\alpha, u\cdot\nabla_x]u\partial^\alpha u\,\mathrm{d}x - \frac{1}{2}\int|\partial^\alpha u|^2\,\operatorname{div}u\,\mathrm{d}x\\
&\quad + \int[-\partial^\alpha, \frac{p'(1+\rho)}{1+\rho}\nabla_x]\rho\partial^\alpha u\,\mathrm{d}x + \int\partial^\alpha\rho\partial^\alpha u\cdot\nabla\frac{p'(1+\rho)}{1+\rho}\,\mathrm{d}x\\
&\quad - \frac{1}{2}\int|\partial^\alpha\rho|^2\operatorname{div}\Big(\frac{p'(1+\rho)}{(1+\rho)^2}u\Big)\,\mathrm{d}x - \int\partial^\alpha(ua)\partial^\alpha u\,\mathrm{d}x\\
&\quad - \sum_{i=1}^3\int\nabla\frac{1}{1+\rho}\nabla\partial^\alpha u_i\partial^\alpha u_i\,\mathrm{d}x - \sum_{1\le\beta\le\alpha}c_{\alpha,\beta}\int\partial^\beta\Big(\frac{1}{1+\rho}\Big)\partial^{\alpha-\beta}\Delta u\partial^\alpha u\,\mathrm{d}x\\
&\quad + \int\langle\partial^\alpha\{\rho(\mathcal{L}f - u\cdot\nabla_v f + u\cdot v\sqrt{M})\}, \partial^\alpha f\rangle\,\mathrm{d}x + \frac{1}{2}\int\partial_t\frac{p'(1+\rho)}{(1+\rho)^2}|\partial^\alpha\rho|^2\,\mathrm{d}x\\
&:= \sum_{j=1}^{14} I_j,
\end{aligned}\tag{4.11}$$

where $C_{\alpha,\beta}$ are constants depending only on α and β. Each term in (4.11) can be estimated as follows. For I_1, I_2, I_5, and I_{10}, we can carry out similar arguments to the proof of Lemma 2.3 in [12] to obtain (the details is omitted here)

$$\begin{aligned}
I_1 &\le C\|\nabla u\|_{H^3}\|\nabla_x f\|_{L^2_v(H^3_x)}\|\nabla_v\partial^\alpha f\|,\\
I_2 &\le C(1+\|\rho\|_{H^4})\|\nabla u\|_{H^3}\|\nabla_x f\|_{L^2_v(H^3_x)}\|v\partial^\alpha f\|,\\
I_5 &\le C\|\nabla u\|^2_{H^3}\|\partial^\alpha u\|,\\
I_{10} &\le C\|\nabla u\|_{H^3}\|\nabla a\|_{H^3}\|\partial^\alpha u\|.
\end{aligned}$$

Using Hölder's, Sobolev's, and Young's inequalities, we easily get the following bounds:

$$\begin{aligned}
I_3 + I_4 &\le C\|\nabla u\|_{H^3}\|\nabla\rho\|_{H^3}\|\partial^\alpha\rho\| \le C\|\nabla u\|_{H^3}\|\nabla\rho\|^2_{H^3},\\
I_6 &\le C\|\operatorname{div}u\|_{L^\infty}\|\partial^\alpha u\|^2 \le C\|\nabla u\|_{H^3}\|\partial^\alpha u\|^2,\\
I_7 &\le C(\|\nabla\rho\|^2_{H^3} + \|\nabla\rho\|^5_{H^3})\|\nabla u\|_{H^3},\\
I_8 &\le C\|u\|_{H^4}\|\nabla\rho\|^2_{H^3},
\end{aligned}$$

$$
\begin{aligned}
I_9 &\le C\big(\|\operatorname{div} u\|_{L^\infty} + \|u\|_{L^\infty}\|\nabla\rho\|_{L^\infty}\big)\|\partial^\alpha\rho\|^2\\
&\le C(1+\|\rho\|_{H^3})\|u\|_{H^3}\|\partial^\alpha\rho\|^2,\\
I_{11} &\le C\|\nabla\rho\|_{H^2}\|\nabla\partial^\alpha u\|\|\partial^\alpha u\| \le C_\epsilon\|\nabla\rho\|_{H^2}^2\|\partial^\alpha u\|^2 + \epsilon\|\nabla\partial^\alpha u\|^2,\\
I_{12} &\le C\|\nabla\rho\|_{H^2}\|\nabla\partial^\alpha u\|\|\partial^\alpha u\| + C\big(\|\nabla\rho\|_{H^3} + \|\nabla\rho\|_{H^3}^4\big)\|\nabla u\|_{H^3}^2\\
&\le C_\epsilon\big(\|\nabla\rho\|_{H^3} + \|\nabla\rho\|_{H^3}^4\big)\|\nabla u\|_{H^3}^2 + \epsilon\|\nabla\partial^\alpha u\|^2
\end{aligned}
$$

with $\epsilon > 0$ a small constant.

By means of (1.7) and (4.1), one has

$$
\sup_{0\le t<T, x\in\mathbb{R}^3}|\partial_t\rho(t,x)| \le (1+\|\rho\|_{L^\infty})\|\operatorname{div} u\|_{L^\infty} + \|u\|_{L^\infty}\|\nabla\rho\|_{L^\infty} \le C\|u\|_{H^3}.
$$

Then, the term I_{14} can be controlled by the following bound

$$
C\|\rho_t\|_{L^\infty}\|\partial^\alpha\rho\|^2 \le C\|u\|_{H^3}\|\partial^\alpha\rho\|^2.
$$

Now we estimate the term I_{13}. We have

$$
\begin{aligned}
\iint -\partial^\alpha(\rho u\cdot\nabla_v f)\partial^\alpha f\,\mathrm{d}x\mathrm{d}v &= \iint \partial^\alpha(\rho u f)\cdot\nabla_v\partial^\alpha f\,\mathrm{d}x\mathrm{d}v\\
&\le C\|\nabla\rho\|_{H^3}\|\nabla u\|_{H^3}\|\nabla_x f\|_{L^2_v(H^3_x)}\|\nabla_v\partial^\alpha f\|,\\
\iint \partial^\alpha(\rho u)\cdot v\sqrt{M}\partial^\alpha f\,\mathrm{d}x\mathrm{d}v &\le C\|\partial^\alpha(\rho u)\|\|v\sqrt{M}\partial^\alpha f\|\\
&\le C\|\nabla\rho\|_{H^3}\|\nabla u\|_{H^3}\|\partial^\alpha f\|_\nu,\\
\iint \partial^\alpha(\rho\mathcal{L}f)\partial^\alpha f\,\mathrm{d}x\mathrm{d}v &= -\iint \partial^\alpha(\rho g)\partial^\alpha g\,\mathrm{d}x\mathrm{d}v\\
&\le C\|\nabla\rho\|_{H^3}\|\nabla g\|_{L^2_v(H^3_x)}^2\\
&\le C\|\nabla\rho\|_{H^3}\sum_{1\le|\alpha'|\le 4}\|\partial^{\alpha'}f\|_\nu^2
\end{aligned}
$$

with $g = \sqrt{M}\nabla_v(M^{-\frac{1}{2}}f)$. Thus, I_{13} can be bounded by

$$
C\big(\|\rho\|_{H^4} + \|\rho\|_{H^4}\|u\|_{H^4}\big)\Big\{\|\nabla_x(a,b,u)\|_{H^3}^2 + \sum_{1\le|\alpha'|\le 4}\|\{\mathbf{I}-\mathbf{P}\}\partial^{\alpha'}f\|_\nu^2\Big\}.
$$

Plugging the estimates on I_i $(1\le i\le 14)$ into (4.11) and taking summation over $1\le|\alpha|\le 4$, we obtain (4.7). □

In order to get the energy dissipation rate $\|\nabla_x(a,b)\|_{H^3}$, we need to study the equations satisfied by a and b. We follow some ideas developed in [12] where the lower order estimates similar to Proposition 4.3 below were obtained. It is easy to verify that a and b satisfy the following equations:

$$
\partial_t a + \operatorname{div} b = 0, \tag{4.12}
$$

$$
\partial_t b_i + \partial_{x_i} a + \sum_j \partial_{x_j}\Gamma_{i,j}(\{\mathbf{I}-\mathbf{P}\}f) = -(1+\rho)b_i + (1+\rho)u_i(1+a), \tag{4.13}
$$

$$
\partial_{x_j} b_i + \partial_{x_i} b_j - (1+\rho)(u_i b_j + u_j b_i) = -\partial_t\Gamma_{i,j}\{\mathbf{I}-\mathbf{P}\}f + \Gamma_{i,j}(l+r+s) \tag{4.14}
$$

for $1 \le i, j \le 3$, where $\Gamma_{i,j}$ is the moment functional defined by

$$\Gamma_{i,j} = \langle (v_i v_j - 1)\sqrt{M}, g\rangle$$

for any $g = g(v)$, and l, r, s are defined respectively by

$$\begin{aligned}
&l := -v \cdot \nabla_x \{\mathbf{I} - \mathbf{P}\}f + \mathcal{L}\{\mathbf{I} - \mathbf{P}\}f,\\
&r := -u \cdot \nabla_v \{\mathbf{I} - \mathbf{P}\}f + \frac{1}{2} u \cdot v \{\mathbf{I} - \mathbf{P}\}f,\\
&s := \rho M^{-\frac{1}{2}} \nabla_v \cdot \Big(\frac{v}{2}\sqrt{M}\{\mathbf{I} - \mathbf{P}\}f + \sqrt{M}\{\mathbf{I} - \mathbf{P}\}f - u\sqrt{M}\{\mathbf{I} - \mathbf{P}\}f\Big).
\end{aligned}$$

In fact, (4.12) and (4.13) can be obtained straightforwardly by multiplying (1.7) by $\sqrt{M}$ and $v_i\sqrt{M}\,(1 \le i \le 3)$ respectively and then taking the velocity integration over $\mathbb{R}^3$. To obtain (4.14), we can rewrite (1.7) as

$$\partial_t \mathbf{P}f + v \cdot \nabla_x \mathbf{P}f + u \cdot \nabla_v \mathbf{P}f - \frac{1}{2} u \cdot v \mathbf{P}f + \mathbf{P}_1 f = -\partial_t \{\mathbf{I} - \mathbf{P}\}f + l + r + s,$$

then apply Γ_{ij} to it and use (4.12).

We have the following estimate.

Proposition 4.3. *For smooth solutions of the problem* (1.7)-(1.10)*, we have*

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_0(t) + \lambda \|\nabla_x (a, b)\|_{H^3}^2 \le\, & C\big(\|\{\mathbf{I} - \mathbf{P}\}f\|^2_{L^2_v(H^4_x)} + \|u - b\|^2_{H^3}\big)\\
&+ C\big(\|\rho, u\|^2_{H^3} + \|\rho\|^2_{H^3}\|u\|^2_{H^3}\big)\\
&\times \big(\|\nabla_x \{\mathbf{I} - \mathbf{P}\}f\|^2_{L^2_v(H^3_x)} + \|u - b\|^2_{H^3} + \|\nabla_x (a, b)\|^2_{H^3}\big)
\end{aligned} \tag{4.15}$$

for all $0 \le t < T$.

Proof. It follows from (4.14) that

$$\begin{aligned}
&\sum_{i,j} \|\partial^\alpha (\partial_i b_j + \partial_j b_i)\|^2\\
&= \sum_{i,j} \int \partial^\alpha (\partial_i b_j + \partial_j b_i)\\
&\quad \times \partial^\alpha [(1 + \rho)(u_i b_j + u_j b_i) - \partial_t \Gamma_{i,j}(\{\mathbf{I} - \mathbf{P}\}f) + \Gamma_{i,j}(l + r + s)]\,\mathrm{d}x\\
&= -\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i,j} \int \partial^\alpha (\partial_i b_j + \partial_j b_i) \partial^\alpha \Gamma_{i,j}(\{\mathbf{I} - \mathbf{P}\}f)\,\mathrm{d}x\\
&\quad + \sum_{i,j} \int \partial^\alpha (\partial_i \partial_t b_j + \partial_j \partial_t b_i) \partial^\alpha \Gamma_{i,j}(\{\mathbf{I} - \mathbf{P}\}f)\,\mathrm{d}x\\
&\quad + \sum_{i,j} \int \partial^\alpha (\partial_i b_j + \partial_j b_i) \partial^\alpha [(1 + \rho)(u_i b_j + u_j b_i) + \Gamma_{i,j}(l + r + s)]\,\mathrm{d}x.
\end{aligned} \tag{4.16}$$

Applying (4.13), Lemma 4.1 and Young's inequality, we obtain

$$\begin{aligned}
&\sum_{i,j} \int \partial^\alpha (\partial_i \partial_t b_j + \partial_j \partial_t b_i) \partial^\alpha \Gamma_{i,j}(\{\mathbf{I} - \mathbf{P}\}f)\,\mathrm{d}x\\
&\qquad \le \epsilon \|\nabla_x a\|^2_{H^3} + C_\epsilon \|\nabla_x \{\mathbf{I} - \mathbf{P}\}f\|^2_{L^2_v(H^3_x)}
\end{aligned}$$

$$+ C(1 + \|\rho\|_{H^3}^2)(\|u - b\|_{H^3}^2 + \|u\|_{H^2}^2 \|\nabla_x a\|_{H^2}^2)$$

with $\epsilon > 0$ sufficiently small. For the final term on the right hand side of (4.16), we have the following estimate:

$$\begin{aligned}\sum_{i,j} \int \partial^\alpha(\partial_i b_j + \partial_j b_i)\partial^\alpha[(1+\rho)(u_i b_j + u_j b_i) + \Gamma_{i,j}(l + r + s)]\,\mathrm{d}x \\ \le \frac{1}{2}\sum_{i,j} \|\partial^\alpha(\partial_i b_j + \partial_j b_i)\|^2 \\ + C\sum_{i,j}(\|\partial^\alpha(1+\rho)(u_i b_j + u_j b_i)\|^2 + \|\partial^\alpha \Gamma_{i,j}(l + r + s)\|^2).\end{aligned}$$

According to Lemma 4.1, the definition of $\Gamma_{i,j}$, and the expressions of l and r, we get

$$\begin{aligned}\sum_{i,j} \|\partial^\alpha(u_i b_j + u_j b_i)\|^2 &\le C(1 + \|\rho\|_{H^3}^2)\|u\|_{H^3}^2 \|\nabla_x b\|_{H^3}^2, \\ \sum_{i,j} \|\partial^\alpha \Gamma_{i,j}(l)\|^2 &\le C\|\{\mathbf{I} - \mathbf{P}\}f\|_{L_v^2(H^4)}^2, \\ \sum_{i,j} \|\partial^\alpha \Gamma_{i,j}(r)\|^2 &\le C\|u\|_{H^3}^2 \|\nabla_x\{\mathbf{I} - \mathbf{P}\}f\|_{L_v^2(H^3)}^2.\end{aligned}$$

For $\Gamma_{i,j}(s)$, we have the following estimate:

$$\Gamma_{i,j}(s) = -\rho \int \{v_i v_j + (v_i \partial_{v_j} + v_i \partial_{v_j}) - (u_i v_j + u_j v_i)\}\sqrt{M}\{\mathbf{I} - \mathbf{P}\}f\mathrm{d}v. \tag{4.17}$$

Now, we deal with the terms on the right hand side of (4.17). First, we have

$$\begin{aligned}\left\|\partial^\alpha \int \rho u_j v_i \sqrt{M}\{\mathbf{I} - \mathbf{P}\}f\mathrm{d}v\right\|^2 &= \int \left|\int v_i \sqrt{M}\partial^\alpha(\rho u_j\{\mathbf{I} - \mathbf{P}\}f)\mathrm{d}v\right|^2 \mathrm{d}x \\ &\le C\int\int |\partial^\alpha(\rho u_j\{\mathbf{I} - \mathbf{P}\}f)|^2 \,\mathrm{d}x\mathrm{d}v \\ &\le C\|\rho\|_{H^3}^2 \|u\|_{H^3}^2 \|\nabla_x\{\mathbf{I} - \mathbf{P}\}f\|_{L_v^2(H_x^3)}^2.\end{aligned}$$

Similarly we obtain that

$$\begin{aligned}\left\|\partial^\alpha \int \rho u_i v_j \sqrt{M}\{\mathbf{I} - \mathbf{P}\}f\mathrm{d}v\right\|^2 &\le C\|\rho\|_{H^3}^2 \|u\|_{H^3}^2 \|\nabla_x\{\mathbf{I} - \mathbf{P}\}f\|_{L_v^2(H_x^3)}, \\ \left\|\partial^\alpha \int \rho v_i v_j \sqrt{M}\{\mathbf{I} - \mathbf{P}\}f\mathrm{d}v\right\|^2 &\le C\|\rho\|_{H^3}^2 \|\nabla_x\{\mathbf{I} - \mathbf{P}\}f\|_{L_v^2(H_x^3)}^2, \\ \left\|\partial^\alpha \int \rho(v_i \partial_{v_j} + v_i \partial_{v_j})\sqrt{M}\{\mathbf{I} - \mathbf{P}\}f\mathrm{d}v\right\|^2 &\le C\|\rho\|_{H^3}^2 \|\nabla_x\{\mathbf{I} - \mathbf{P}\}f\|_{L_v^2(H_x^3)}^2.\end{aligned}$$

Notice that

$$\sum_{i,j} \|\partial^\alpha(\partial_i b_j + \partial_j b_i)\|^2 = 2\|\nabla_x \partial^\alpha b\|^2 + 2\|\nabla_x \cdot \partial^\alpha b\|^2.$$

Using the above equality and plugging the above estimates into (4.16), and then taking summation over $|\alpha| \leq 3$, one gets

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\alpha|\leq 3}\sum_{i,j}\int \partial^\alpha(\partial_i b_j+\partial_j b_i)\partial^\alpha\Gamma_{i,j}(\{\mathbf{I}-\mathbf{P}\}f)\,\mathrm{d}x+\|\nabla_x\partial^\alpha b\|^2+\|\nabla_x\cdot\partial^\alpha b\|^2\\
\leq\,&\epsilon\|\nabla a\|_{H^3}^2+C_\epsilon\|\{\mathbf{I}-\mathbf{P}\}f\|_{L_x^2(H_x^4)}^2+C\|u-b\|_{H^3}^2+C\big(\|(\rho,u)\|_{H^3}^2+\|\rho\|_{H^3}^2\|u\|_{H^3}^2\big)\\
&\times\big(\|\nabla_x\{\mathbf{I}-\mathbf{P}\}f\|_{L_v^2(H_x^3)}^2+\|u-b\|_{H^3}^2+\|\nabla_x(a,b)\|_{H^3}^2\big). \qquad (4.18)
\end{aligned}$$

On the other hand, by means of (4.12) and (4.13), it follows that

$$\begin{aligned}
\|\partial^\alpha\nabla a\|^2=&-\frac{\mathrm{d}}{\mathrm{d}t}\int\partial^\alpha a\partial^\alpha\mathrm{div}\,b\,\mathrm{d}x+\|\partial^\alpha\mathrm{div}\,b\|^2\\
&+\sum_i\int\partial^\alpha\partial_i a\partial^\alpha\Big\{(1+\rho)(u_i-b_i)-\sum_j\partial_j\Gamma_{i,j}\{\mathbf{I}-\mathbf{P}\}f+(1+\rho)u_i a\Big\}\,\mathrm{d}x\\
\leq&-\frac{\mathrm{d}}{\mathrm{d}t}\int\partial^\alpha a\partial^\alpha\mathrm{div}\,b\,\mathrm{d}x+\|\partial^\alpha\mathrm{div}\,b\|^2+\frac{1}{2}\|\partial^\alpha\nabla a\|^2+C\|\nabla_x\{\mathbf{I}-\mathbf{P}\}f\|_{L_v^2(H_x^3)}^2\\
&+C(1+\|\rho\|_{H^3}^2)\big\{\|u-b\|_{H^3}^2+\|u\|_{H^3}^2\|\nabla a\|_{H^3}^2\big\}. \qquad (4.19)
\end{aligned}$$

Summing (4.19) over $|\alpha|\leq 3$ and taking $\epsilon=\frac{1}{4}$, and then adding the results into (4.18) implies (4.15). ☐

Proposition 4.4. *For smooth solutions of the problem* (1.7)-(1.10)*, we have*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\alpha|\leq 3}\int_{\mathbb{R}^3}\partial^\alpha u\cdot\partial^\alpha\nabla\rho\,\mathrm{d}x+\lambda\|\nabla\rho\|_{H^3}^2\\
&\quad\leq C(\|u-b\|_{H^3}^2+\|\nabla u\|_{H^4}^2)+C(\|\rho\|_{H^4}+\|\rho\|_{H^4}^8+\|u\|_{H^3}^2)\|\nabla(\rho,u)\|_{H^3}^2 \qquad (4.20)
\end{aligned}$$

for all $0\leq t<T$.

Proof. Taking differentiation ∂^α ($|\alpha|\leq 3$) to (1.9), and by carrying an direct calculation, we get

$$\begin{aligned}
p'(1)\|\nabla\partial^\alpha\rho\|^2=&-\int\nabla\partial^\alpha\rho\partial^\alpha\partial_t u\,\mathrm{d}x+\int\nabla\partial^\alpha\rho\partial^\alpha(b-u)\,\mathrm{d}x\\
&+\int\nabla\partial^\alpha\rho\partial^\alpha\Big\{-u\cdot\nabla u+\frac{1}{1+\rho}\Delta u-\Big[\frac{p'(1+\rho)}{1+\rho}-p'(1)\Big]\nabla\rho\Big\}\,\mathrm{d}x\\
:=&\,Q_1+Q_2+Q_3. \qquad (4.21)
\end{aligned}$$

For Q_i ($i=1,2,3$), applying (1.8), Hölder's, Sobolev's and Young's inequalities, we have

$$\begin{aligned}
Q_1=&-\frac{\mathrm{d}}{\mathrm{d}t}\int\nabla\partial^\alpha\rho\partial^\alpha u\,\mathrm{d}x+\int\partial^\alpha\ \mathrm{div}\,u\partial^\alpha[(1+\rho)\ \mathrm{div}\,u+u\cdot\nabla\rho]\,\mathrm{d}x\\
\leq&-\frac{\mathrm{d}}{\mathrm{d}t}\int\nabla\partial^\alpha\rho\partial^\alpha u\,\mathrm{d}x+C\|\partial^\alpha\ \mathrm{div}\,u\|^2+C\|\rho\|_{H^4}\|\nabla u\|_{H^3}^2,\\
Q_2\leq&\frac{p'(1)}{4}\|\nabla\partial^\alpha\rho\|^2+C\|u-b\|_{H^3}^2,\\
Q_3\leq&\frac{p'(1)}{8}\|\nabla\partial^\alpha\rho\|^2+C\|u\cdot\nabla u\|_{H^3}^2+C\Big\|\frac{1}{1+\rho}\Delta u\Big\|_{H^3}^2
\end{aligned}$$

$$
\begin{aligned}
&+C\Big\|\Big[\frac{p'(1+\rho)}{1+\rho}-p'(1)\Big]\nabla\rho\Big\|_{H^3}^2\\
\le&\frac{p'(1)}{4}\|\nabla\partial^\alpha\rho\|^2+C\|u\|_{H^3}^2\|\nabla u\|_{H^3}^2+C\|\rho\|_{H^3}\|\nabla\partial^\alpha\rho\|^2+C\|\nabla u\|_{H^4}^2\\
&+C(\|\rho\|_{H^3}^2+\|\rho\|_{H^3}^8)\|(\nabla\rho,\nabla u)\|_{H^3}^2.
\end{aligned}
$$

With the help of (4.1), plugging the above estimates into (4.21), we obtain (4.20). □

Reorganizing the estimates obtained above in the Propositions 4.1-4.4, one gets

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1(t)+\lambda\mathcal{D}_1(t)\\
\le&C\big\{\|(\rho,u)\|_{H^4}+\|\rho\|_{H^4}\|u\|_{H^4}(1+\|\rho\|_{H^4}^2+\|\rho\|_{H^4}\|u\|_{H^4})+\|\rho\|_{H^4}(1+\|\rho\|_{H^4}^7)\big\}\\
&\times\Big\{\|\nabla(a,b,\rho,u)\|_{H^3}^2+\|b-u\|_{H^4}^2+\sum_{|\alpha|\le4}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2\Big\}.
\end{aligned}
\tag{4.22}
$$

Next, we need to estimate the mixed space-velocity derivatives of f, i.e., $\partial_\beta^\alpha f$. Since $\|\partial_\beta^\alpha\mathbf{P}f\|\le C\|\partial^\alpha f\|$ for any α and β, we only need to estimate $\|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\|$ below. Let us apply $\mathbf{I}-\mathbf{P}$ to both sides of (1.7) to get

$$
\begin{aligned}
&\partial_t\{\mathbf{I}-\mathbf{P}\}f+v\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}f+u\cdot\nabla_v\{\mathbf{I}-\mathbf{P}\}f-\frac{1}{2}u\cdot v\{\mathbf{I}-\mathbf{P}\}f\\
&\quad=\mathcal{L}\{\mathbf{I}-\mathbf{P}\}f+\mathbf{P}\Big\{v\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}f+u\cdot\nabla_v\{\mathbf{I}-\mathbf{P}\}f-\frac{1}{2}u\cdot v\{\mathbf{I}-\mathbf{P}\}f\Big\}\\
&\qquad-\{\mathbf{I}-\mathbf{P}\}\Big\{v\cdot\nabla_x\mathbf{P}f+u\cdot\nabla_v\mathbf{P}f-\frac{1}{2}u\cdot v\mathbf{P}f\Big\}+\{\mathbf{I}-\mathbf{P}\}G.
\end{aligned}
\tag{4.23}
$$

where $\{\mathbf{I}-\mathbf{P}\}G$ is defined by

$$
\begin{aligned}
\{\mathbf{I}-\mathbf{P}\}G:=\rho\Big\{&\mathcal{L}\{\mathbf{I}-\mathbf{P}\}f+\frac{1}{2}u\cdot v\{\mathbf{I}-\mathbf{P}\}f-u\cdot\nabla_v\{\mathbf{I}-\mathbf{P}\}f\\
&+\mathbf{P}\Big(u\cdot\nabla_v\{\mathbf{I}-\mathbf{P}\}f-\frac{1}{2}u\cdot v\{\mathbf{I}-\mathbf{P}\}f\Big)\\
&-\{\mathbf{I}-\mathbf{P}\}\Big(u\cdot\nabla_v\mathbf{P}f-\frac{1}{2}u\cdot v\mathbf{P}f\Big)\Big\}.
\end{aligned}
$$

Now we give a simple derivation of the equality (4.23). First,

$$
\begin{aligned}
\{\mathbf{I}-\mathbf{P}\}(v\cdot\nabla_x f)&=v\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}f+v\cdot\mathbf{P}f-\mathbf{P}(v\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}f)-\mathbf{P}(v\cdot\nabla_x\mathbf{P}f)\\
&=v\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}f+\{\mathbf{I}-\mathbf{P}\}(v\cdot\nabla_x\mathbf{P}f)-\mathbf{P}(v\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}f).
\end{aligned}
$$

We can deal with the terms $\{\mathbf{I}-\mathbf{P}\}(u\cdot\nabla_v f),\{\mathbf{I}-\mathbf{P}\}(u\cdot vf)$ in the same way. Meanwhile, by the definitions of $\mathcal{L}$ and $\mathbf{P}$, one has

$$
\{\mathbf{I}-\mathbf{P}\}(u\cdot v\sqrt{M})\equiv0,\qquad\{\mathbf{I}-\mathbf{P}\}\mathcal{L}f=\mathcal{L}\{\mathbf{I}-\mathbf{P}\}f.
$$

Then (4.23) follows.

In the proof of the following Proposition, we adopt some ideas from [16, Lemma 4.3].

Proposition 4.5. *Let* $1 \le k \le 4$. *For smooth solutions of the problem* (1.7)-(1.10), *we have*

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\substack{|\beta|=k\\|\alpha|+|\beta|\le 4}}\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2+\lambda\sum_{\substack{|\beta|=k\\|\alpha|+|\beta|\le 4}}\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2\\
&\quad\le C(1+\|(\rho,u)\|_{H^3}^2+\|\rho\|_{H^3}^2\|u\|_{H^3}^2)\sum_{|\alpha'|\le 4-k+1}\|\partial^{\alpha'}\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2\\
&\qquad+C(\|\rho\|_{H^3}+\|u\|_{H^3}^2+\|\rho\|_{H^3}^2\|u\|_{H^3}^2)\sum_{\substack{1\le|\beta'|\le 4\\|\alpha'|+|\beta'|\le 4}}\|\partial^{\alpha'}_{\beta'}\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2\\
&\qquad+C\|\nabla b\|_{H^{4-k}}^2+C(1+\|\rho\|_{H^3}^2)\|u\|_{H^{4-k}}^2\|\nabla b\|_{H^3}^2\\
&\qquad+C\chi_{2\le k\le 4}(1+\|\rho\|_{H^3}^2)\sum_{\substack{1\le|\beta'|\le k-1\\|\alpha'|+|\beta'|\le 4}}\|\partial^{\alpha'}_{\beta'}\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2
\end{aligned}\tag{4.24}
$$

for all $0 \le t < T$. *Here* χ_E *denotes the characteristic function of the set* E.

Proof. Fix $k\,(1 \le k \le 4)$. Choosing α and β such that $|\beta| = k$ and $|\alpha| + |\beta| \le 4$, multiplying (4.23) by $\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f$ and then taking integration, one has

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2+\int\langle -L\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f,\{\mathbf{I}-\mathbf{P}\}f\rangle\,\mathrm{d}x:=\sum_{i=1}^{7}J_i \tag{4.25}
$$

with

$$
\begin{aligned}
J_1&=\int\langle -\partial^\alpha_x[\partial^\beta_v,v\cdot\nabla_x]\{\mathbf{I}-\mathbf{P}\}f,\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\rangle\,\mathrm{d}x,\\
J_2&=\int\langle \partial^\alpha_x[\partial^\beta_v,-|v|^2]\{\mathbf{I}-\mathbf{P}\}f,\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\rangle\,\mathrm{d}x,\\
J_3&=\int\langle -\partial^\alpha_\beta(u\cdot\nabla_v\{\mathbf{I}-\mathbf{P}\}f),\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\rangle\,\mathrm{d}x,\\
J_4&=\int\langle \frac{1}{2}\partial^\alpha_\beta(u\cdot v\{\mathbf{I}-\mathbf{P}\}f),\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\rangle\,\mathrm{d}x,\\
J_5&=\int\Big\langle \partial^\alpha_\beta\mathbf{P}\Big(v\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}f+u\cdot\nabla_v\{\mathbf{I}-\mathbf{P}\}f\\
&\qquad\qquad-\frac{1}{2}u\cdot v\{\mathbf{I}-\mathbf{P}\}f\Big),\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\Big\rangle\,\mathrm{d}x,\\
J_6&=\int\Big\langle -\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}\Big(v\cdot\nabla_x\mathbf{P}f+u\cdot\nabla_v\mathbf{P}f-\frac{1}{2}u\cdot v\mathbf{P}f\Big),\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\Big\rangle\,\mathrm{d}x,\\
J_7&=\int\langle \partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}G,\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\rangle\,\mathrm{d}x.
\end{aligned}
$$

Here the fact that $[\partial^\beta_v,\mathcal{L}]=[\partial^\beta_v,-|v|^2]$ has been used.

Now we start estimating each term J_i in (4.25). For the terms $J_i\,(i=1,\cdots,6)$, we have

$$
J_1\le\eta\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2+C_\eta\|[\partial^\beta_v,v\cdot\nabla_v]\partial^\alpha_x\{\mathbf{I}-\mathbf{P}\}f\|^2
$$

$$
\begin{aligned}
&\leq \eta\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2 + C_\eta \sum_{|\alpha'|\leq 4-k} \|\partial^{\alpha'}\nabla_x\{\mathbf{I}-\mathbf{P}\}f\|^2 \\
&\quad + \chi_{2\leq k\leq 4} C_\eta \sum_{\substack{1\leq|\beta'|\leq k-1\\ |\alpha'|+|\beta'|\leq 4}} \|\partial^{\alpha'}_{\beta'}\{\mathbf{I}-\mathbf{P}\}f\|^2,\\
J_2 &\leq \eta\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2 + C_\eta\|[\partial^\beta_v, -|v|^2]\partial^\alpha_x\{\mathbf{I}-\mathbf{P}\}f\|^2\\
&\leq \eta\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2 + C_\eta \sum_{|\alpha'|\leq 4-k} \|\partial^{\alpha'}\{\mathbf{I}-\mathbf{P}\}f\|^2_\nu \\
&\quad + \chi_{2\leq k\leq 4} C_\eta \sum_{\substack{1\leq|\beta'|\leq k-1\\ |\alpha'|+|\beta'|\leq 4}} \|\partial^{\alpha'}_{\beta'}\{\mathbf{I}-\mathbf{P}\}f\|^2_\nu,\\
J_3 &\leq \eta\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2 + C_\eta\|\partial^\alpha_x(u\cdot\nabla_v\partial^\beta_v\{\mathbf{I}-\mathbf{P}\}f)\|^2\\
&\leq \eta\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2 + C_\eta\|u\|^2_{H^3} \sum_{\substack{1\leq|\beta'|\leq 4\\ |\alpha'|+|\beta'|\leq 4}} \|\partial^{\alpha'}_{\beta'}\{\mathbf{I}-\mathbf{P}\}f\|^2,\\
J_4 &\leq \eta\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2 + C_\eta\|\partial^\alpha_x\big(u\cdot\partial^\beta_v(v\{\mathbf{I}-\mathbf{P}\}f)\big)\|^2\\
&\leq \eta\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2 + C_\eta\|u\|^2_{H^3} \sum_{\substack{1\leq|\beta'|\leq 4\\ |\alpha'|+|\beta'|\leq 4}} \|\partial^{\alpha'}_{\beta'}\{\mathbf{I}-\mathbf{P}\}f\|^2_\nu\\
&\quad + C_\eta\|u\|^2_{H^3}\sum_{|\alpha'|\leq 4-k}\|\partial^{\alpha'}\{\mathbf{I}-\mathbf{P}\}f\|^2,\\
J_5 &\leq \eta\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2 + C_\eta\|\partial^\alpha_\beta\mathbf{P}(v\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}f)\|^2\\
&\quad + C_\eta\|\partial^\alpha_\beta\mathbf{P}(u\cdot\nabla_v\{\mathbf{I}-\mathbf{P}\}f)\|^2 + C_\eta\|\partial^\alpha_\beta\mathbf{P}(u\cdot v\{\mathbf{I}-\mathbf{P}\}f)\|^2\\
&\leq \eta\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2 + C_\eta\sum_{|\alpha'|\leq 4-k}\|\nabla_x\partial^{\alpha'}_x\{\mathbf{I}-\mathbf{P}\}f\|^2\\
&\quad + C_\eta\|u\|^2_{H^3}\sum_{|\alpha'|\leq 4-k}\|\partial^{\alpha'}_x\{\mathbf{I}-\mathbf{P}\}f\|^2,\\
J_6 &\leq \eta\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2 + C_\eta\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}(v\cdot\nabla_x\mathbf{P}f)\|^2\\
&\quad + C_\eta\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}(u\cdot\nabla_v\mathbf{P}f)\|^2 + C_\eta\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}(u\cdot v\mathbf{P}f)\|^2\\
&\leq \eta\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2 + C_\eta\big(\|\nabla b\|^2_{H^{4-k}} + \|\nabla b\|^2_{H^2}\|u\|^2_{H^{4-k}}\big).
\end{aligned}
$$

For the term J_7, we have the following calculation and estimates:

$$
\begin{aligned}
J_7 = &\int\langle\partial^\alpha_\beta\big(\rho L\{\mathbf{I}-\mathbf{P}\}f\big), \partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\rangle\,\mathrm{d}x\\
&+\frac{1}{2}\int\langle\partial^\alpha_\beta\big(\rho u\cdot v\{\mathbf{I}-\mathbf{P}\}f\big), \partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\rangle\,\mathrm{d}x\\
&-\int\langle\partial^\alpha_\beta\big(\rho u\cdot\nabla_v\{\mathbf{I}-\mathbf{P}\}f\big), \partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\rangle\,\mathrm{d}x\\
&+\int\Big\langle\partial^\alpha_\beta\mathbf{P}\Big(\rho u\cdot\nabla_v\{\mathbf{I}-\mathbf{P}\}f - \frac{1}{2}\rho u\cdot v\{\mathbf{I}-\mathbf{P}\}f\Big), \partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\Big\rangle\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&-\int\Big\langle \partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}\Big(\rho u\cdot\nabla_v\mathbf{P}f-\frac{1}{2}\rho u\cdot v\mathbf{P}f\Big),\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\Big\rangle\,\mathrm{d}x\\
:=&\sum_{i=1}^{5}Y_i.
\end{aligned}
$$

We can adopt the above similar estimates to deal with $Y_i\,(2\le i\le 5)$. Thus, we only give the following bounds:

$$
\begin{aligned}
Y_2\le\,&\eta\|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2\\
&+C_\eta\|\rho\|_{H^3}^2\|u\|_{H^3}^2\Big\{\sum_{|\alpha'|\le 4-k}\|\partial^{\alpha'}\{\mathbf{I}-\mathbf{P}\}f\|^2+\sum_{\substack{1\le|\beta'|\le 4\\|\alpha'|+|\beta'|\le 4}}\|\partial_{\beta'}^{\alpha'}\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2\Big\},\\
Y_3\le\,&\eta\|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2+C_\eta\|\rho\|_{H^3}^2\|u\|_{H^3}^2\sum_{\substack{1\le|\beta'|\le 4\\|\alpha'|+|\beta'|\le 4}}\|\partial_{\beta'}^{\alpha'}\{\mathbf{I}-\mathbf{P}\}f\|^2,\\
Y_4\le\,&\eta\|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2+C_\eta\|\rho\|_{H^3}^2\|u\|_{H^3}^2\sum_{|\alpha'|\le 4-k}\|\partial^{\alpha'}\{\mathbf{I}-\mathbf{P}\}f\|^2,\\
Y_5\le\,&\eta\|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2+C_\eta\|\rho\|_{H^3}^2\|u\|_{H^{4-k}}^2\|\nabla_x b\|_{H^2}^2.
\end{aligned}
$$

For Y_1, we give a detailed calculation:

$$
\begin{aligned}
Y_1=&\iint\partial_x^\alpha\big(\rho L\partial_v^\beta\{\mathbf{I}-\mathbf{P}\}f\big)\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\,\mathrm{d}x\mathrm{d}v\\
&+\iint\partial_x^\alpha\big(\rho[\partial_v^\beta,-|v|^2]\{\mathbf{I}-\mathbf{P}\}f\big)\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\,\mathrm{d}x\mathrm{d}v,\\
=&\iint\rho L\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\,\mathrm{d}x\mathrm{d}v\\
&+\sum_{1\le\gamma\le\alpha}C_{\alpha,\gamma}\iint\partial^\gamma\rho L\partial_\beta^{\alpha-\gamma}\{\mathbf{I}-\mathbf{P}\}f\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\,\mathrm{d}x\mathrm{d}v\\
&+\iint\partial_x^\alpha\big(\rho[\partial_v^\beta,-|v|^2]\{\mathbf{I}-\mathbf{P}\}f\big)\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\,\mathrm{d}x\mathrm{d}v,\\
:=&Y_{11}+Y_{12}+Y_{13}.
\end{aligned}
$$

For Y_{11}, we can move it to the left hand side of the equality (4.25). Thus, we only need to deal with Y_{12} and Y_{13}. We have

$$
\begin{aligned}
Y_{12}=&-\iint\partial^\gamma\rho\sqrt{M}\nabla_v\big(M^{-\frac{1}{2}}\partial_\beta^{\alpha-\gamma}\{\mathbf{I}-\mathbf{P}\}f\big)\sqrt{M}\nabla_v\big(\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\big)\,\mathrm{d}x\mathrm{d}v\\
\le&\,C\|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\nu\|\rho\|_{H^3}\sum_{\substack{|\beta'|=k\\|\alpha'|+|\beta'|\le 4}}\|\partial_{\beta'}^{\alpha'}\{\mathbf{I}-\mathbf{P}\}f\|_\nu\\
\le&\,C\|\rho\|_{H^3}\sum_{\substack{|\beta'|=k\\|\alpha'|+|\beta'|\le 4}}\|\partial_{\beta'}^{\alpha'}\{\mathbf{I}-\mathbf{P}\}f\|_\nu^2,\\
Y_{13}\le&\,\eta\|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2+C_\eta\|[\partial_v^\beta,-|v|^2]\partial_x^\alpha\rho\{\mathbf{I}-\mathbf{P}\}f\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \eta\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2 + C_\eta\|\rho\|^2_{H^3}\Big\{\sum_{|\alpha'|\leq 4-k}\|\partial^{\alpha'}\{\mathbf{I}-\mathbf{P}\}f\|^2_\nu \\
&\quad + \chi_{2\leq k\leq 4}\sum_{\substack{1\leq|\beta'|\leq k-1\\ |\alpha'|+|\beta'|\leq 4}}\|\partial^{\alpha'}_{\beta'}\{\mathbf{I}-\mathbf{P}\}f\|^2_\nu\Big\}.
\end{aligned}
$$

For the second term in the left hand side of the equality (4.25), one gets

$$
\begin{aligned}
&\int(1+\rho)\langle -L\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f, \partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\rangle\,\mathrm{d}x \\
&\geq \lambda_1\|\{\mathbf{I}-\mathbf{P}_0\}f\|^2_\nu \\
&\geq \frac{\lambda_1}{2}\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2_\nu - \lambda_1\|\mathbf{P}_0\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2_\nu \\
&\geq \frac{\lambda_1}{2}\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2_\nu - C\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2.
\end{aligned}
$$

Plugging all the above estimates into (4.25) and choosing η sufficiently small, we obtain (4.24). □

Remark 4.1. According to the above Lemma, we can choose some suitable constants C_k, such that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{1\leq k\leq 4}C_k\sum_{\substack{|\beta|=k\\ |\alpha|+|\beta|\leq 4}}\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2 + \lambda\sum_{\substack{1\leq|\beta|\leq 4\\ |\alpha|+|\beta|\leq 4}}\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2_\nu \\
&\leq C(\|(\rho,u)\|^2_{H^3} + \|\rho\|^2_{H^3}\|u\|^2_{H^3})\sum_{|\alpha|\leq 4}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2_\nu \\
&\quad + C(\|\rho\|_{H^3} + \|u\|^2_{H^3} + \|\rho\|^2_{H^3}\|u\|^2_{H^3})\sum_{\substack{1\leq||\beta|\leq 4\\ \alpha|+|\beta|\leq 4}}\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|^2_\nu \\
&\quad + C\|u\|^2_{H^3}(1+\|\rho\|^2_{H^3})\|\nabla b\|^2_{H^3} + C\Big(\|\nabla b\|^2_{H^3} + \sum_{|\alpha|\leq 4}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|^2_\nu\Big). \qquad (4.26)
\end{aligned}
$$

With the aid of the inequalities (4.22), (4.26) and (4.1), we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + \lambda\mathcal{D}(t) \leq C\big(\mathcal{E}^{\frac{1}{2}}(t) + \mathcal{E}^2(t)\big)\mathcal{D}(t) \leq C(\delta+\delta^2)\mathcal{D}(t). \qquad (4.27)
$$

So, as long as $0<\delta<1$ is sufficiently small, the integration in time of (4.27) yields

$$
\mathcal{E}(t) + \lambda\int_0^t\mathcal{D}(s)\mathrm{d}s \leq \mathcal{E}(0) \qquad (4.28)
$$

for all $0\leq t<T$. Besides, (4.1) can be justified by choosing

$$
\mathcal{E}(0) \sim \|f_0\|^2_{H^4_{x,v}} + \|(\rho_0,u_0)\|^2_{H^4}
$$

sufficiently small.

4.2. **The case of periodic domain.** In this subsection we deal with the uniform a priori estimate when Ω is a spatial periodic domain $\mathbb{T}^3$. Using the following conservation laws in the case of torus,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\iint F\,\mathrm{d}x\mathrm{d}v=0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\int n\,\mathrm{d}x=0,\\ &\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\int nu\,\mathrm{d}x+\iint vF\,\mathrm{d}x\mathrm{d}v\Big\}=0, \end{aligned}$$

and by the assumptions of Theorem 1.2, it follows that

$$\int a\,\mathrm{d}x=0, \quad \int \rho\,\mathrm{d}x=0, \quad \int \big(b+(1+\rho)u\big)\,\mathrm{d}x=0 \tag{4.29}$$

for all $t\geq 0$.

Thanks to the Poincaré inequality and the conservation laws (4.29), we have

$$\|a\|_{L^2}\leq C\|\nabla a\|_{L^2}, \quad \|\rho\|_{L^2}\leq C\|\nabla\rho\|_{L^2}, \tag{4.30}$$

$$\begin{aligned} \|u+b\|_{L^2} &\leq \|b+u+\rho u\|_{L^2}+\|\rho u\|_{L^2}\\ &\leq C\|\nabla(b+u+\rho u)\|_{L^2}+\|u\|_{L^\infty}\|\rho\|_{L^2}\\ &\leq C\|\nabla(b,u)\|_{L^2}+C\|u\|_{H^2}\|\nabla\rho\|_{L^2}+C\|\rho\|_{H^2}\|\nabla u\|_{L^2}. \end{aligned} \tag{4.31}$$

Similarly to the whole space case, we can obtain the following estimates:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1(t)+\lambda\mathcal{D}_1(t)\leq C(\mathcal{E}_1^{\frac12}+\mathcal{E}_1^2)\mathcal{D}_1(t), \tag{4.32}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_2(t)+\lambda\mathcal{D}_2(t)\leq C\mathcal{D}_1(t)+C(\mathcal{E}_1+\mathcal{E}_1^2)\mathcal{D}_1(t)+C(\mathcal{E}_1^{\frac12}+\mathcal{E}_1^2)\mathcal{D}_2(t). \tag{4.33}$$

According to the definition of $\mathcal{D}_{\mathbb{T},1}(t)$, we have

$$\mathcal{D}_{\mathbb{T},1}(t)\sim\sum_{|\alpha|\leq 4}\|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha f\|_\nu^2+\|(a,b,\rho,u)\|_{H^4}^2. \tag{4.34}$$

Combining (4.30), (4.31) and (4.32) together, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1(t)+\lambda\mathcal{D}_{\mathbb{T},1}(t)\leq C(\mathcal{E}_1^{\frac12}+\mathcal{E}_1^2)\mathcal{D}_{\mathbb{T},1}(t). \tag{4.35}$$

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